Synchronization of multiple fractional order multi-scroll Chen chaotic systems using polynomial fuzzy controller

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Abstract

In this paper, synchronization of multiple integer order and fractional order multi-scroll Chenchaotic systems through polynomial fuzzy modeling using polynomial fuzzy control method is presented. For synchronization of multiple chaotic systems, fuzzy control is designed to ensure that multiple response systems are synchronized with one excitation system. Sum of squares methodis used to find feedback gains of the polynomial controller. Finally, simulation results show effectiveness and high efficiency of the proposed method.

Keywords: fractional order chaotic systems, synchronization, polynomial fuzzy modelling, sum of squares (SOS)

1. Introduction

Synchronization of multiple chaotic systemshas attracted attentions recently [1]. This issue has a promising future for multiple communication and other engineering areas both in theory and applications [2-5]. Various synchronization schemes have been proposed for multiple chaotic systems. Like complete synchronization [6-8], anti-synchronization [9], projective synchronization, ring synchronization [10] and hybrid synchronization [11, 12]. Two types of synchronizations have been presented to link multiple chaotic systems. One of the multiple response systems is synchronized with slave systems. In another case, ring transmission synchronization mode is performed among multiple systems with a ring connection [13-15]. These cases are applied successfully in hybrid and information engineering networks. Various synchronization schemes have been developed in the literature like linear and nonlinear feedback control [6,7], direct design method [8-10], pulse control [16], sample data control [17], link control [18].

Most studies performed on synchronization of multiple systems have not used polynomial fuzzy model. In this study, the main aim is synchronization of multiple chaotic systems through polynomial fuzzy modelling.

A fuzzy model is a mathematical tool for representing a nonlinear system. This fuzzy model is then used to analyze and design a controller. Various fuzzy models have been presented. Among these models, linear and polynomial type-I and type-II fuzzy sets can be mentioned.

Although T-S fuzzy system has shown various advantages, but it has some disadvantages. Considering nonlinearity concept, a mathematical model can be described as a T-S fuzzy model by considering acompact agent area. Then, T-S fuzzy model can be considered as a local nonlinear model. Analytical results of the T-S fuzzy model are valid when the system operates in the operational area of interest. Results obtained from LMIs are very conservative and consequent section of the T-S can be written only as a linear matrix. This problem can be resolved by shifting linear subsystems in the consequent section of rules using polynomial systems. This model is called type-I polynomial fuzzy controller. Despite existence of polynomials in subsystems, this fuzzy model can model a wider range of nonlinear systems. In the polynomial fuzzy system, the fuzzy controller is based on sum of squares(SOS) and polynomial fuzzy modelling is more effective than fuzzy controller based on LMI [19]. Consequent section of the polynomial fuzzy model is comprised of polynomial matrix. Therefore, stability condition of SOS is more general than stability condition of LMI [20].

Therefore, polynomial fuzzy systems give more simple measures for analyzing stability and designing controllers. Number of local models is generally less than T-S fuzzy systems and SOS design conditions consider design methods based on LMI for T-S fuzzy model as a particular case.

Error dynamic state feedback fuzzy controllers have dynamics at their output which is determined using a set of first order differential equations. This type of fuzzy controllers is used to track reference control and reject disturbance.

In this study, first, Chen master chaotic system and two slave systems are transferred to equivalent polynomial fuzzy models. Then, a polynomial fuzzy controller is designed to synchronize master and slave polynomial fuzzy models. Polynomial fuzzy controller is designed considering SOS conditions which can be done using SOS optimization tool called SOSTOOLS [21].

Fractional order calculus has been used in many areas including electronics, mechanics, electrical engineering [22-25]. Recently, it has been proved that fractional order differential equations are better than integer order for modelling many physical phenomena in science and engineering [26]. Fractional order chaotic systems might have more useful applications compared to integer order systems. Chaotic behavior has been observed in several fractional order systems like Chen [27], Lorenz [28], Lu [29] and Rossler [30]. Synchronization of fractional order systems has increased among researchers due to their potential applications [31-34]. Considering previous studies, fractional order multi-state synchronization has been considered in a few studies [35]. Considering the above discussion, in this paper, polynomial fuzzy modelling and synchronization of multiple fractional order multi-scroll Chen systems is considered. According to the previous studies, synchronization of multiple fractional order multi-scroll chaotic Chen systems has not been considered.

The rest of this paper is organized as follows. Section 2 presents model of the integer order system and problem formulation. Section 3 describes fractional order chaotic system model and fractional order formulation. Section 4 presents simulation results. Finally, section 5 concludes the paper.

2. System Description and Formulating Synchronization of Multiple Integer Oder Chaotic Chen Systems

A multi-scroll chaotic Chen system is considered as follows:

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = (c - a - x_3 + d\sin x_3)x_1 + cx_2 \quad (1) \\ \dot{x}_3 = x_1 x_2 - bx_3 \end{cases}$$

Where x_1, x_2, x_3 are state variables and a, b, c and d are system parameters. Such that $a = 35, b = 3, c = 28, d = 8$. A Chen multi-scroll attractor is shown in Figure 1.

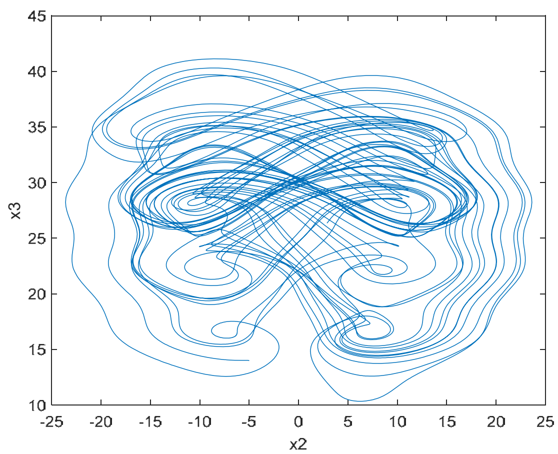

Figure 1. Phase curve of x3,x2 of the Chen multi-scroll chaotic system

In order to present master and slave synchronization problem, master slave is considered as follows.
Master chaotic system:

$$\begin{cases} \dot{x}_{m1} = a(x_{m2} - x_{m1}) \\ \dot{x}_{m2} = (c - a - x_{m3} + d\sin x_{m3})x_{m1} + cx_{m2} \quad (3) \\ \dot{x}_{m3} = x_{m1} x_{m2} - bx_{m3} \end{cases}$$

First slave chaotic system:

$$\begin{cases} \dot{x}_{s1} = a(x_{s2} - x_{s1}) + u_{11} \\ \dot{x}_{s2} = (c - a - x_{s3} + d \sin x_{s3})x_{s1} + cx_{s2} + u_{12} \\ \dot{x}_{s3} = x_{s1}x_{s2} - bx_{s3} + u_{13} \end{cases} \quad (4)$$

Second slave chaotic system:

$$\begin{cases} \dot{y}_{s1} = a(x_{s2} - x_{s1}) + u_{21} \\ \dot{y}_{s2} = (c - a - x_{s3} + d \sin x_{s3})x_{s1} + cx_{s2} + u_{22} \\ \dot{y}_{s3} = x_{s1}x_{s2} - bx_{s3} + u_{23} \end{cases} \quad (5)$$

where x_{m1}, x_{m2}, x_{m3} are state variables of the master chaotic system and x_{s1}, x_{s2}, x_{s3} are state variables of the first slave chaotic system and y_{s1}, y_{s2}, y_{s3} are state variables of the second slave chaotic system. u_{11}, u_{12}, u_{13} and u_{21}, u_{22}, u_{23} are control inputs of the first and second slave chaotic systems which should be designed to synchronize master and slave systems.

2.1. Design of Polynomial Fuzzy Control for Synchronizing Integer Order Chaotic Systems

In this section, design of polynomial fuzzy control is presented for synchronization of master chaotic system (3) and slave chaotic systems (4) and (5). In this design, master and slave chaotic systems are transferred to equivalent polynomial fuzzy models. Then, a polynomial fuzzy model is designed for synchronization of polynomial fuzzy master and slave models which are equivalent to main master and slave chaotic systems.

A. Polynomial fuzzy model of the chaotic system

Master chaotic system (3) and slave chaotic systems (4) and (5) can be represented accurately as polynomial fuzzy models:

$$x_m = \sum_{i=1}^{2} h_i(x_m) A_i(x_m) x_m \quad (6)$$

$$x_s = \sum_{i=1}^{2} h_i(x_s) A_i(x_s) x_s + u_1 \quad (7)$$

$$y_s = \sum_{i=1}^{2} h_i(y_s) A_i(y_s) y_s + u_2 \quad (8)$$

Where $x_m = \begin{bmatrix} x_{m1} & x_{m2} & x_{m3} \end{bmatrix}^T$ is state vector of the master system, $x_s = \begin{bmatrix} x_{s1} & x_{s2} & x_{s3} \end{bmatrix}^T$ and $y_s = \begin{bmatrix} y_{s1} & y_{s2} & y_{s3} \end{bmatrix}^T$ are state vectors of the first and second slave systems and $u = [u_1 \quad u_2 \quad u_3]$ is the control input. $h_i(x)$ is the normalized membership degree, $A_i(x) \in \Re^{n \times n}$ is system matrix where n is dimension of the system equal to3.

And

$$h_1(x_m) = \begin{cases} \dfrac{6(\sin x_{m3} - x_{m3})}{x_{m3}^3} + 1 & x_{m3} \neq 0 \\ 0, & x_{m3} = 0 \end{cases}$$

$$h_2(x_m) = 1 - h_1(x_m)$$

$$h_1(x_s) = \begin{cases} \dfrac{6(\sin x_{s3} - x_{s3})}{x_{s3}^3} + 1 & x_{s3} \neq 0 \\ 0, & x_{s3} = 0 \end{cases}$$

$$h_2(x_s) = 1 - h_1(x_s)$$

$$h_1(y_s)=\begin{cases}\dfrac{6(\sin y_{s3}-y_{s3})}{y_{s3}^3}+1 & y_{s3}\neq 0\\ 0, & y_{s3}=0\end{cases}$$

$$h_2(y_s)=1-h_1(y_s)$$

$$A_1(x_m)=\begin{bmatrix}-a & a & 0\\ c-a-x_{m3} & c & dx_{m1}\\ x_{m2} & 0 & -b\end{bmatrix}$$

$$A_2(x_m)=\begin{bmatrix}-a & a & 0\\ c-a-x_{m3} & c & dx_{m1}(1-x_{m3}^2/6)\\ x_{m2} & 0 & -b\end{bmatrix}$$

$$A_1(x_s)=\begin{bmatrix}-a & a & 0\\ c-a-x_{s3} & c & dx_{s1}\\ x_{s2} & 0 & -b\end{bmatrix}$$

$$A_2(x_s)=\begin{bmatrix}-a & a & 0\\ c-a-x_{s3} & c & dx_{s1}(1-x_{s3}^2/6)\\ x_{s2} & 0 & -b\end{bmatrix}$$

$$A_1(y_s)=\begin{bmatrix}-a & a & 0\\ c-a-y_{s3} & c & dy_{s1}\\ y_{s2} & 0 & -b\end{bmatrix}$$

$$A_2(y_s)=\begin{bmatrix}-a & a & 0\\ c-a-y_{s3} & c & dx_{s1}(1-y_{s3}^2/6)\\ y_{s2} & 0 & -b\end{bmatrix}$$

B. Design of a Polynomial Active Fuzzy Controller

$e1\equiv x_s-x_m$ and $e2\equiv y_s-x_m$ are defined and a polynomial fuzzy controller is designed to synchronize master chaotic system (3) and slave chaotic systems (4) and (5) as follows:

$u_1=u_{n1}+u_{f1}$ (7)

with

$u_{n1}=-\sum_{i=1}^{2}h_i(x_s)A_i(x_s)x_s+\sum_{i=1}^{2}h_i(x_m)A_i(x_m)x_m$ (8)

$u_{f1}=-(\sum_{i=1}^{2}h_i(x_m)F_i(x_m))e1.$ (9)

$u_2=u_{n2}+u_{f2}$ (10)

with

$u_{n2}=-\sum_{i=1}^{2}h_i(y_s)A_i(y_s)y_s+\sum_{i=1}^{2}h_i(x_m)A_i(x_m)x_m$ (11)

$$u_{f2} = -(\sum_{i=1}^{2} h_i(x_m)F_i(x_m))e2. \quad (12)$$

Theorem 1. Master chaotic system (3) and slave chaotic systems (4) and (5) can be synchronized using polynomial fuzzy controller (7) and (8). If there exists a definite matrix $X \in R^{3*3}$ and polynomial matrices $M_i(x_m) \in R^{3*3}$ such that the following conditions are met:

$$v^T(X - \varepsilon_1 I)v \quad is \quad SOS \quad (13)$$

$$v^T(M_i^T(x_m) + M_i(x_m) - \varepsilon_2 I)v \quad is \quad SOS \quad i = 1,2 \quad (14)$$

Where ε_1 and ε_2 are small positive values. In this case, feedback gain in (9) and (12) is obtained as follows:

$$F_i(x_m) = M_i(x_m)X^{-1} \quad (15)$$

Proof. Chaotic system (3) and slave chaotic system (4) are transferred to polynomial fuzzy models (5) and (6). Therefore, system error can be defined as follows:

$$\begin{aligned} e1 &= x_s - x_m \\ e2 &= y_s - x_m \end{aligned} \quad (16)$$

$$\begin{aligned} \dot{e}1 &= \dot{x}_s - \dot{x}_m = \sum_{i=1}^{2} h_i(x_s)A_i(x_s)x_s + u1 - \sum_{i=1}^{2} h_i(x_m)A_i(x_m)x_m \\ \dot{e}2 &= \dot{y}_s - \dot{x}_m = \sum_{i=1}^{2} h_i(y_s)A_i(y_s)y_s + u2 - \sum_{i=1}^{2} h_i(x_m)A_i(x_m)x_m \end{aligned} \quad (17)$$

Lyapunov function is defined as follows:

$$V(e1,e2) = P\sum_{i=1}^{3} e1_i^2 + P\sum_{i=1}^{3} e2_i^2 \quad (18)$$

Where $P = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = X^{-1}$. Then, derivative of V is obtained as follows:

$$\begin{aligned} \dot{V}(e) &= e_{11}(\dot{e}_{11}) + e_{12}\dot{e}_{12} + e_{13}\dot{e}_{13} + e_{21}(\dot{e}_{21}) + e_{22}\dot{e}_{22} + e_{23}\dot{e}_{23} = \\ & e_{11}([\sum_{i=1}^{2} h_i(x_{s1})A_i(x_{s1})x_{s1}] - \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1}] + u_{11}) + \\ & e_{12}([\sum_{i=1}^{2} h_i(x_{s2})A_i(x_{s2})x_{s2}] - \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2}] + u_{12}) + \\ & e_{13}([\sum_{i=1}^{2} h_i(x_{s3})A_i(x_{s3})x_{s3}] - \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3}] + u_{13}) + \\ & e_{21}([\sum_{i=1}^{2} h_i(y_{s1})A_i(y_{s1})y_{s1}] - \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1}] + u_{21}) + \\ & e_{22}([\sum_{i=1}^{2} h_i(y_{s2})A_i(y_{s2})y_{s2}] - \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2}] + u_{22}) + \\ & e_{23}([\sum_{i=1}^{2} h_i(y_{s3})A_i(y_{s3})y_{s3}] - \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3}] + u_{23}) \end{aligned} \quad (19)$$

If

$$
\begin{aligned}
u_{11} &= -\sum_{i=1}^{2} h_i(x_{s1})A_i(x_{s1})x_{s1} + \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1} + u_{f11} \\
u_{12} &= -\sum_{i=1}^{2} h_i(x_{s2})A_i(x_{s2})x_{s2} + \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2} + u_{f12} \\
u_{13} &= -\sum_{i=1}^{2} h_i(x_{s3})A_i(x_{s3})x_{s3} + \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3} + u_{f13} \\
u_{21} &= -\sum_{i=1}^{2} h_i(y_{s1})A_i(y_{s1})y_{s1} + \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1} + u_{f21} \\
u_{22} &= -\sum_{i=1}^{2} h_i(y_{s2})A_i(y_{s2})y_{s2} + \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2} + u_{f22} \\
u_{23} &= -\sum_{i=1}^{2} h_i(y_{s3})A_i(y_{s3})y_{s3} + \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3} + u_{f23}
\end{aligned} \qquad (20)
$$

Then

$$\dot{V} = e_{11}u_{f11} + e_{12}u_{f12} + e_{13}u_{f13} + e_{21}u_{f21} + e_{22}u_{f22} + e_{23}u_{f23} \quad (21)$$

By substituting u_f, we have:

$$
\begin{aligned}
\dot{V} = &-\sum_{i=1}^{2} h_i(x_{m1})F_i(x_{m1})e_{11}^2 - \sum_{i=1}^{2} h_i(x_{m2})F_i(x_{m2})e_{12}^2 - \sum_{i=1}^{2} h_i(x_{m3})F_i(x_{m3})e_{13}^2 \\
&-\sum_{i=1}^{2} h_i(x_{m1})F_i(x_{m1})e_{21}^2 - \sum_{i=1}^{2} h_i(x_{m2})F_i(x_{m2})e_{22}^2 - \sum_{i=1}^{2} h_i(x_{m3})F_i(x_{m3})e_{23}^2 \qquad (22) \\
&\leq 0
\end{aligned}
$$

It is assumed that k>0.

By applying (15) to (22), we have:

$$
\begin{aligned}
&-(PM_i(x_m)^T)P + P(-M_i(x_m)P) < 0 \\
&\Rightarrow M_i(x_m)^T + M_i(x_m) > 0, i = 1,2,3 \qquad (23)
\end{aligned}
$$

Inequality (23) is established if (13) and (14) are held. It is assumed that k>0. Since $F_i(x_m)$ in (15) is positive definite SOS, $\dot{V} < 0$ is held.

3. System Description and Formulating Multi-State Synchronization of Fractional Order Chaotic Chen System

Fractional order chaotic Chen system is represented in (24):

$$
\begin{cases}
D^q x_1 = a(x_2 - x_1) \\
D^q x_2 = (c - a - x_3 + d \sin x_3)x_1 + cx_2 \\
D^q x_3 = x_1 x_2 - bx_3
\end{cases} \qquad (24)
$$

In which D^q is derivative with fractional order of q where $q \in (0,1)$. The system is simulated using Graunwald-Letinkov definition of differentiation.Graunwald-Letinkon definition of differentiation is as follows [35].

$$D_t^q f(t) = \lim_{t \to \infty} \frac{1}{h^q} \sum_{j=0}^{\infty} (-1)^j \binom{q}{j} f(t - jh) \quad (25)$$

Where $\binom{q}{j} = \frac{q!}{j!(q-j)!} = \frac{q!}{j!(q-j)!} = \frac{\Gamma(q+1)}{\Gamma(j+1)\Gamma(q-j+1)}$

If $n = \dfrac{t-a}{h}$ where a is a real constant describing a limit value and the following can be written:

$$ {}_aD_t^q = \lim_{h\to 0} \frac{1}{h^q}(-1)^j \binom{q}{j} f(t-jh) \quad (26)$$

In numerical calculation of fractional order derivatives, numerical approximation of qth derivative at kh points (k=1,2,…) preserves the following state:

$$(k - L_m / h) D_{t_k}^q f(t) \approx h^{-q} \sum_{j=0}^{k} (-1)^j \binom{q}{j} f(t_k - j) \quad (27)$$

Using Grunwald-Letinkov definition, a fractional order system can be defined as follows:

$$D^q x(t) = f(x,t) \quad (28)$$

In order to analyze stability of systems similar to (4), a fractional order expansion of Lyapunov direct method has been presented in [36] which is given in Theorem 1.

Theorem 1. (Fractional order expansion of Lyapunov direct method). Assume that y=0 is a balance point for non-autonomous fractional order system. Assume that a Lyapunov function V(t,x(t)) and K-class functions $\delta_i = (i = 1,2,3)$ are held:

$$\delta_1(\|x\|) \le V(t,x(t)) \le \delta_1(\|x\|) \quad (29)$$

$$D_t^q V(t,x(t)) \le \delta_3(\|x\|) \quad (30)$$

Where $q \in (0,1)$. Then, system (3) is asymptotically stable.

Phase curve of fractional order Chen system with q=0.99 is shown in Figure 2.

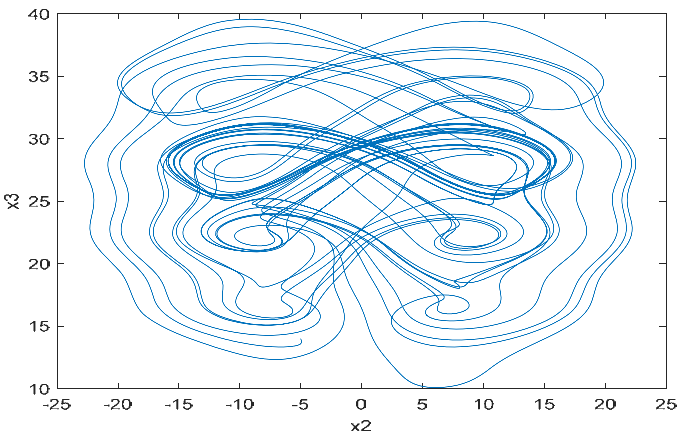

Figure 2. Phase curve of x2,x3 of the fractional order Chen system

In order to present master and slave synchronization problem, the master system is considered as follows:
Master chaotic system:

$$\begin{cases} D^q x_{m1} = a(x_{m2} - x_{m1}) \\ D^q \dot{x}_{m2} = (c - a - x_{m3} + d \sin x_{m3}) x_{m1} + c x_{m2} \quad (31) \\ D^q \dot{x}_{m3} = x_{m1} x_{m2} - b x_{m3} \end{cases}$$

First slave chaotic system:

$$\begin{cases} D^q x_{s1} = a(x_{s2} - x_{s1}) + u_{11} \\ D^q \dot{x}_{s2} = (c - a - x_{s3} + d \sin x_{s3}) x_{s1} + c x_{s2} + u_{12} \\ D^q \dot{x}_{s3} = x_{s1} x_{s2} - b x_{s3} + +u_{13} \end{cases} \quad (32)$$

Second slave chaotic system:

$$\begin{cases} D^q y_{s1} = a(x_{s2} - x_{s1}) + u_{21} \\ D^q y_{s2} = (c - a - x_{s3} + d \sin x_{s3}) x_{s1} + c x_{s2} + u_{22} \\ D^q y_{s3} = x_{s1} x_{s2} - b x_{s3} + u_{23} \end{cases} \quad (33)$$

Where x_{m1}, x_{m2}, x_{m3} are state variables of the master chaotic system and x_{s1}, x_{s2}, x_{s3} are state variables of the first slave chaotic system and y_{s1}, y_{s2}, y_{s3} are state variables of the second slave chaotic system. u_{11}, u_{12}, u_{13} and u_{21}, u_{22}, u_{23} are control inputs of the first and second slave chaotic systems which should be designed to synchronize master and slave systems.

3.2. Design of a Polynomial Fuzzy Control for Synchronization of Fractional Order Chaotic System

In this section, polynomial fuzzy control is designed for synchronization of master chaotic system (31) and slave chaotic system (32) and (33). In the following, polynomial fuzzy model of the chaotic system is described.

A. Polynomial fuzzy model of the chaotic system

Master chaotic system (31) and slave chaotic systems (32) and (33) can be represented accurately as polynomial fuzzy models:

$$D^q x_m = \sum_{i=1}^{2} h_i(x_m) A_i(x_m) x_m \quad (34)$$

$$D^q x_s = \sum_{i=1}^{2} h_i(x_s) A_i(x_s) x_s + u_1 \quad (35)$$

$$D^q y_s = \sum_{i=1}^{2} h_i(y_s) A_i(y_s) y_s + u_2 \quad (36)$$

Where $x_m = [x_{m1} \quad x_{m2} \quad x_{m3}]^T$ is state vector of the master system, $x_s = [x_{s1} \quad x_{s2} \quad x_{s3}]^T$ and $y_s = [y_{s1} \quad y_{s2} \quad y_{s3}]^T$ are state vectors of the first and second slave systems and $u = [u_1 \quad u_2 \quad u_3]$ is the control input. $h_i(x)$ is the normalized membership degree, $A_i(x) \in \Re^{n \times n}$ is system matrix where n is dimension of the system which is 3 and its structure is similar to the one described for integer order system.

B. Design of a Polynomial Active Fuzzy Controller

$e1 \equiv x_s - x_m$ and $e2 \equiv y_s - x_m$ are defined and a polynomial fuzzy controller is designed to synchronize master chaotic system (31) and slave chaotic systems (32) and (33) as follows:

$$u_1 = u_{n1} + u_{f1} \quad (37)$$

with

$$u_{n1} = -\sum_{i=1}^{2} h_i(x_s) A_i(x_s) x_s + \sum_{i=1}^{2} h_i(x_m) A_i(x_m) x_m \quad (38)$$

$$u_{f1} = -\left(\sum_{i=1}^{2} h_i(x_m) F_i(x_m)\right) e1. \quad (39)$$

$$u_2 = u_{n2} + u_{f2} \quad (40)$$

with

$$u_{n2} = -\sum_{i=1}^{2} h_i(y_s)A_i(y_s)y_s + \sum_{i=1}^{2} h_i(x_m)A_i(x_m)x_m \quad (41)$$

$$u_{f2} = -(\sum_{i=1}^{2} h_i(x_m)F_i(x_m))e2. \quad (42)$$

Theorem 2. Master chaotic system (31) and slave chaotic system (32) and (33) can be synchronized using polynomial fuzzy controller (37) and (40) if there exists a definite matrix $X \in R^{3*3}$ and polynomial matrices $M_i(x_m) \in R^{3*3}$ such that the following conditions are met:

$$v^T(X - \varepsilon_1 I)v \quad is \quad SOS \quad (43)$$

$$v^T(M_i^T(x_m) + M_i(x_m) - \varepsilon_2 I)v \quad is \quad SOS \quad i = 1,2 \quad (44)$$

Where ε_1 and ε_2 are small positive values. In this case, feedback gain in (9) and (12) is obtained as follows:

$$F_i(x_m) = M_i(x_m)X^{-1} \quad (45)$$

Proof. Chaotic system (31) and slave chaotic systems (32) and (33) are transferred to polynomial fuzzy models (34), (35) and (36). Therefore, system error can be defined as follows:

$$\begin{aligned} e1 &= x_s - x_m \\ e2 &= y_s - x_m \end{aligned} \quad (46)$$

$$\begin{aligned} D^q e1 &= \dot{x}_s - \dot{x}_m = \sum_{i=1}^{2} h_i(x_s)A_i(x_s)x_s + u1 - \sum_{i=1}^{2} h_i(x_m)A_i(x_m)x_m \\ D^q e2 &= \dot{y}_s - \dot{x}_m = \sum_{i=1}^{2} h_i(y_s)A_i(y_s)y_s + u2 - \sum_{i=1}^{2} h_i(x_m)A_i(x_m)x_m \end{aligned} \quad (47)$$

Lyapunov function is defined as follows:

$$V(e1,e2) = P\sum_{i=1}^{3} e1_i^2 + P\sum_{i=1}^{3} e2_i^2 \quad (48)$$

Where $P = X^{-1}$. Then, derivative of V is obtained as follows:

$$\begin{aligned} \dot{V}(e) &= e_{11}(D^q e_{11}) + e_{12}D^q e_{12} + e_{13}D^q e_{13} + e_{21}(D^q e_{21}) + e_{22}D^q e_{22} + e_{23}D^q e_{23} = \\ & e_{11}([\sum_{i=1}^{2} h_i(x_{s1})A_i(x_{s1})x_{s1}] - \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1}] + u_{11}) + \\ & e_{12}([\sum_{i=1}^{2} h_i(x_{s2})A_i(x_{s2})x_{s2}] - \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2}] + u_{12}) + \\ & e_{13}([\sum_{i=1}^{2} h_i(x_{s3})A_i(x_{s3})x_{s3}] - \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3}] + u_{13}) + \\ & e_{21}([\sum_{i=1}^{2} h_i(y_{s1})A_i(y_{s1})y_{s1}] - \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1}] + u_{21}) + \\ & e_{22}([\sum_{i=1}^{2} h_i(y_{s2})A_i(y_{s2})y_{s2}] - \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2}] + u_{22}) + \\ & e_{23}([\sum_{i=1}^{2} h_i(y_{s3})A_i(y_{s3})y_{s3}] - \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3}] + u_{23}) \end{aligned} \quad (49)$$

If

$$
\begin{aligned}
u_{11} &= -\sum_{i=1}^{2} h_i(x_{s1})A_i(x_{s1})x_{s1} + \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1} + u_{f\,11} \\
u_{12} &= -\sum_{i=1}^{2} h_i(x_{s2})A_i(x_{s2})x_{s2} + \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2} + u_{f\,12} \\
u_{13} &= -\sum_{i=1}^{2} h_i(x_{s3})A_i(x_{s3})x_{s3} + \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3} + u_{f\,13} \\
u_{21} &= -\sum_{i=1}^{2} h_i(y_{s1})A_i(y_{s1})y_{s1} + \sum_{i=1}^{2} h_i(x_{m1})A_i(x_{m1})x_{m1} + u_{f\,21} \\
u_{22} &= -\sum_{i=1}^{2} h_i(y_{s2})A_i(y_{s2})y_{s2} + \sum_{i=1}^{2} h_i(x_{m2})A_i(x_{m2})x_{m2} + u_{f\,22} \\
u_{23} &= -\sum_{i=1}^{2} h_i(y_{s3})A_i(y_{s3})y_{s3} + \sum_{i=1}^{2} h_i(x_{m3})A_i(x_{m3})x_{m3} + u_{f\,23}
\end{aligned} \quad (50)
$$

Then

$$\dot{V} = e_{11}u_{f\,11} + e_{12}u_{f\,12} + e_{13}u_{f\,13} + e_{21}u_{f\,21} + e_{22}u_{f\,22} + e_{23}u_{f\,23} \quad (51)$$

By substituting u_f, we have:

$$
\begin{aligned}
\dot{V} = &-\sum_{i=1}^{2} h_i(x_{m1})F_i(x_{m1})e_{11}^2 - \sum_{i=1}^{2} h_i(x_{m2})F_i(x_{m2})e_{12}^2 - \sum_{i=1}^{2} h_i(x_{m3})F_i(x_{m3})e_{13}^2 \\
&-\sum_{i=1}^{2} h_i(x_{m1})F_i(x_{m1})e_{21}^2 - \sum_{i=1}^{2} h_i(x_{m2})F_i(x_{m2})e_{22}^2 - \sum_{i=1}^{2} h_i(x_{m3})F_i(x_{m3})e_{23}^2 \quad (52) \\
&\leq= 0
\end{aligned}
$$

It is assumed that k>0.

By applying (45) to (52), we have:

$$
\begin{aligned}
&-(PM_i(x_m)^T)P + P(-M_i(x_m)P) < 0 \\
&\Rightarrow M_i(x_m)^T + M_i(x_m) > 0, i = 1,2,3
\end{aligned} \quad (53)
$$

Inequality (53) is established if (43) and (44) are held. It is assumed that k>0. Since $F_i(x_m)$ in (45) is positive definite SOS, $\dot{V} < 0$ is held.

4. Simulation and Results

4.1. Simulation of Integer Order

In numerical simulations, initial condition of the master system is $(x_{m1}(0), x_{m2}(0), x_{m3}(0)) = (-9,-5,14)$, first slave system is $(x_{s1}(0), x_{s2}(0), x_{s3}(0)) = (2,7,-10)$ and second slave system is $(y_{s1}(0), y_{s2}(0), y_{s3}(0)) = (-5,-3,3)$. Total simulation time is 6s and time step is 0.001s. Figures 3-5 and 6-8 show synchronization between states of master and first slave system and synchronization between master and second slave system. Figures 9 and 10 show synchronization error between master and first slave system and master and second slave system, respectively.

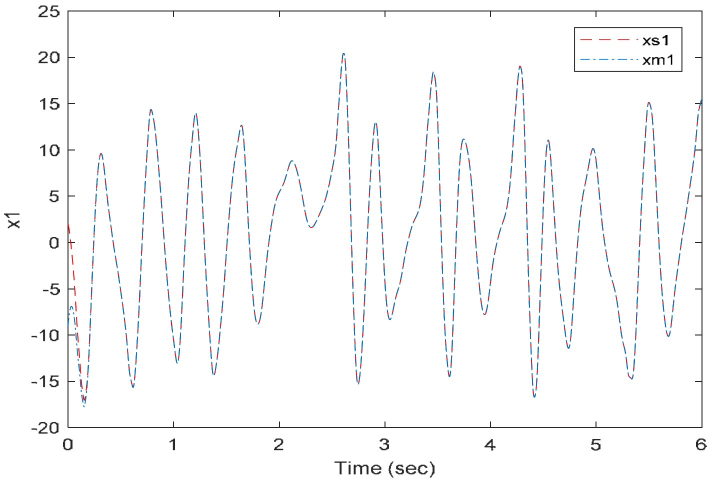

Figure 3. Time curve of (x_{m1}, x_{s1}) of systems (3) and (4)

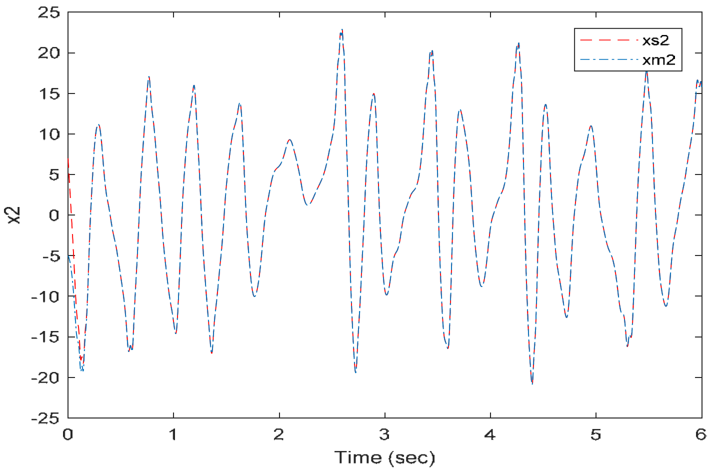

Figure 4. Time curve of (x_{m2}, x_{s2}) of systems (3) and (4)

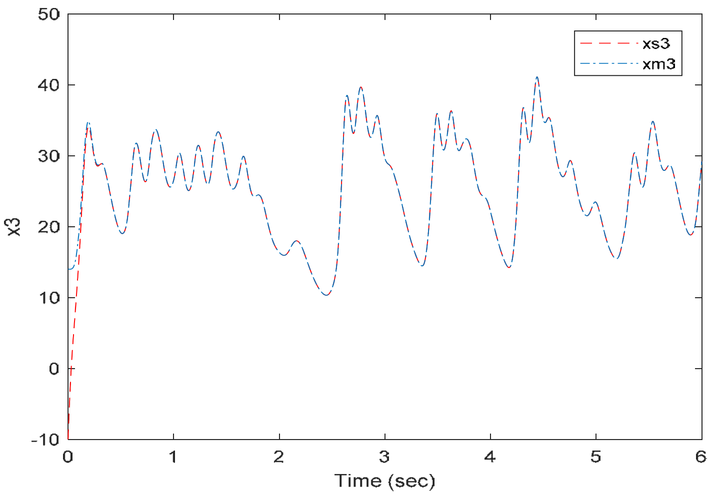

Figure 5. Time curve of (x_{m3}, x_{s3}) of systems (3) and (4)

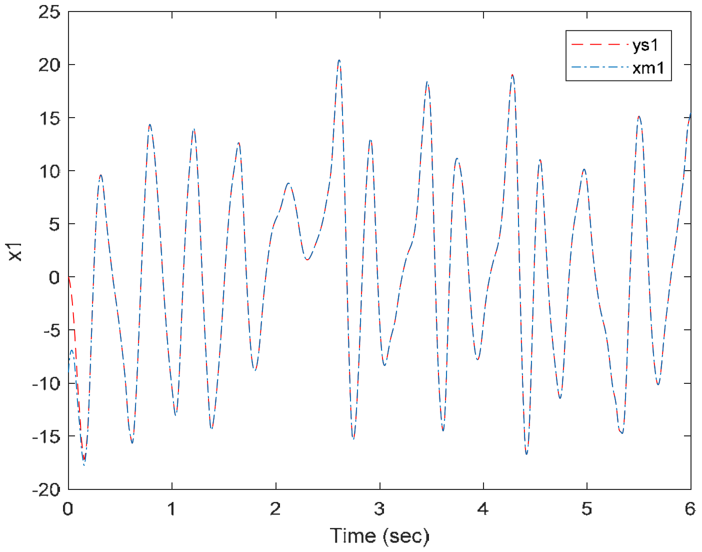

Figure 6. Time curve of (x_{m1}, y_{s1}) of systems (3) and (4)

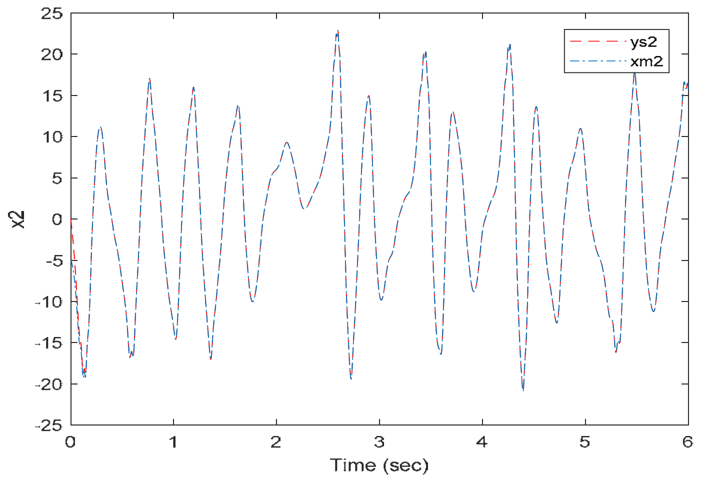

Figure 7. Time curve of (x_{m2}, y_{s2}) of systems (3) and (4)

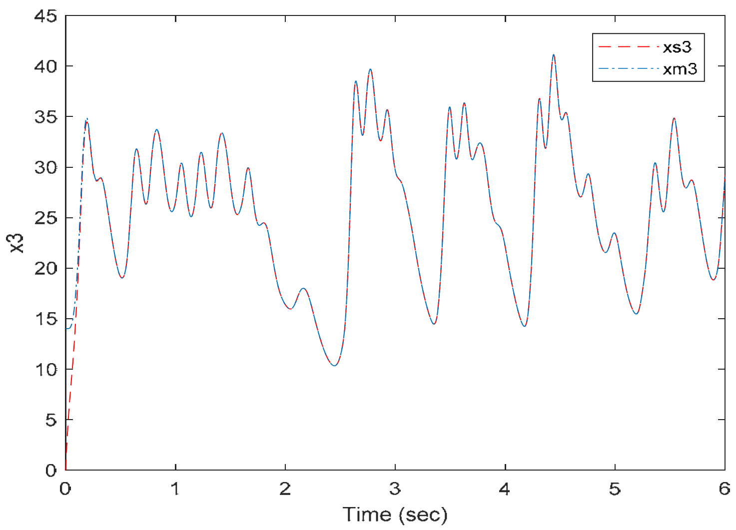

Figure 8. Time curve of (x_{m3}, y_{s3}) of systems (3) and (4)

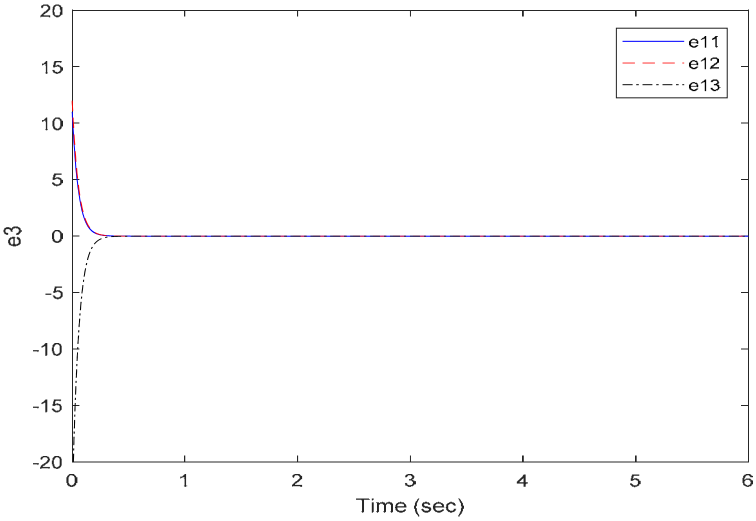

Figure 9. Synchronization error curve between two systems (3) and (4)

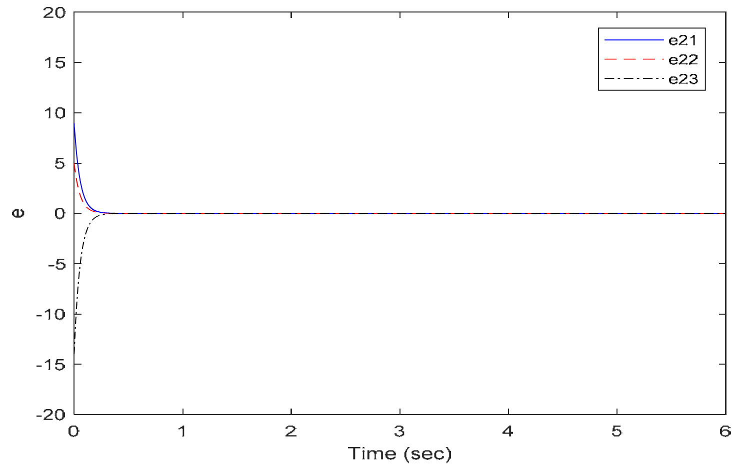

Figure 10. Synchronization error curve between two systems (3) and (4)

It can be seen that synchronization errors converge to zero fast. It is also seen that synchronization is done in 0.3s.

4.2. Synchronization of Multiple Multi-Scroll Chaotic Chen Systemsusing Fractional Order Polynomial Fuzzy Model

In this case, fractional order value of the master and the first slave system and the second slave system is 0.97 and 0.94, respectively. Initial condition and parameters are similar to the integer order model. Time step is 0.001. Simulation is performed for synchronization of multiple systems. Figures 11-13 show synchronization of the master and the first slave system and figures 14-16 shows synchronization of the master and the second slave system. Synchronization error for these two cases is shown in Figures 17 and 18.

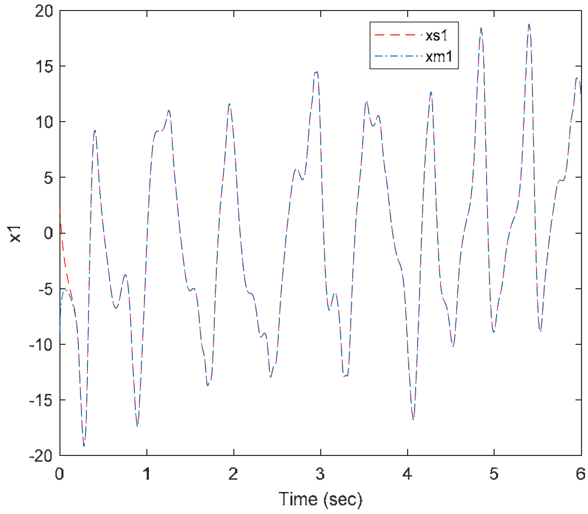

Figure 11. Time curve of (x_{m1}, x_{s1}) of systems (27) and (28)

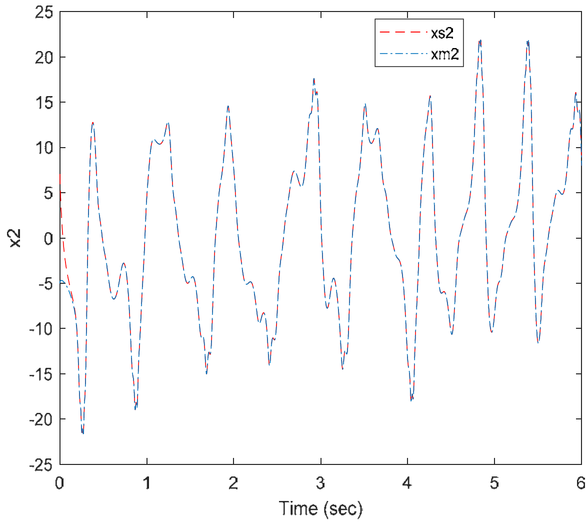

Figure 12. Time curve of (x_{m2}, x_{s2}) of systems (27) and (28)

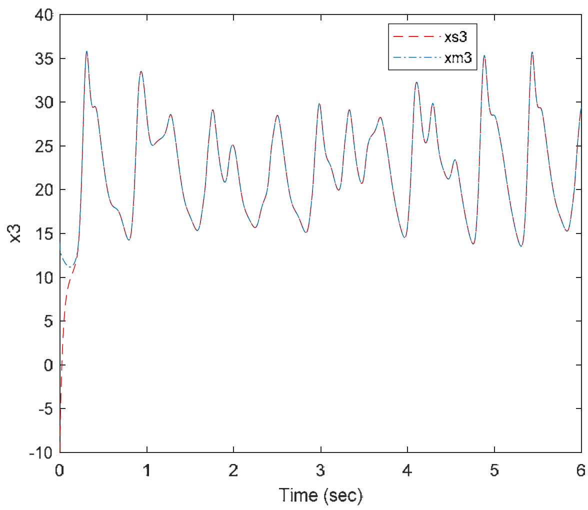

Figure 13. Time curve of (x_{m3}, x_{s3}) of systems (27) and (28)

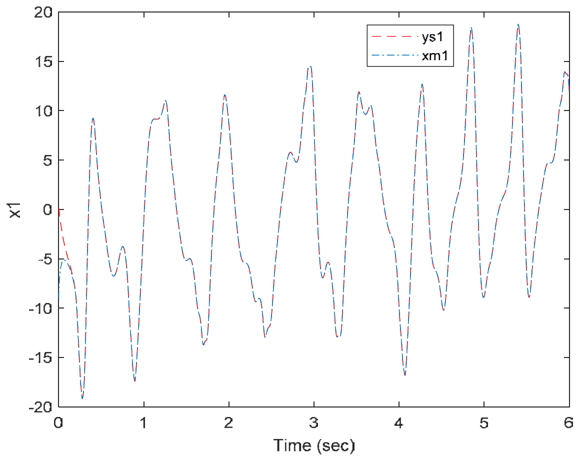

Figure 14. Time curve of (x_{m1}, y_{s1}) of systems (27) and (28)

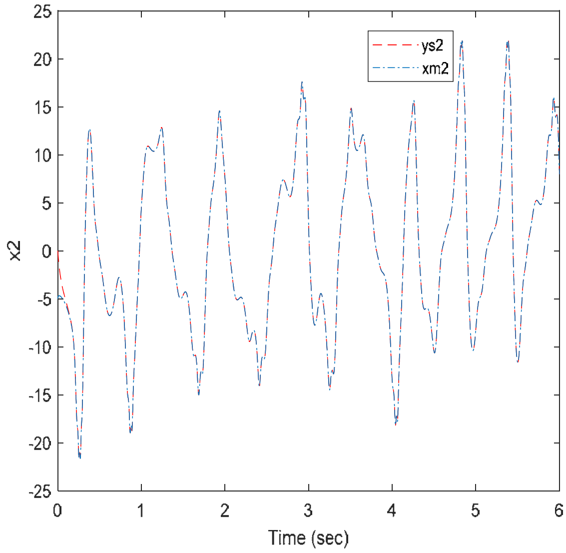

Figure 15. Time curve of (x_{m2}, y_{s2}) of systems (27) and (28)

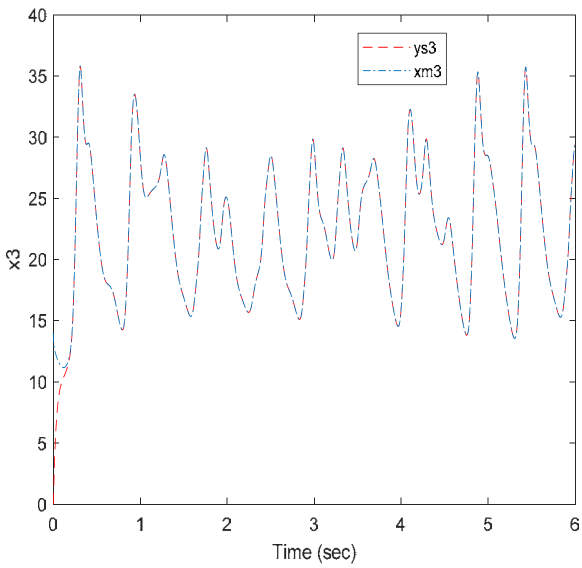

Figure 16. Time curve of (x_{m3}, y_{s3}) of systems (27) and (28)

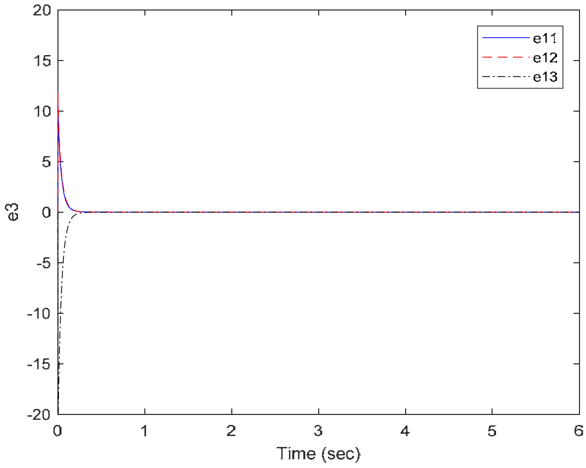

Figure 17. Synchronization error curve between two systems (27) and (28)

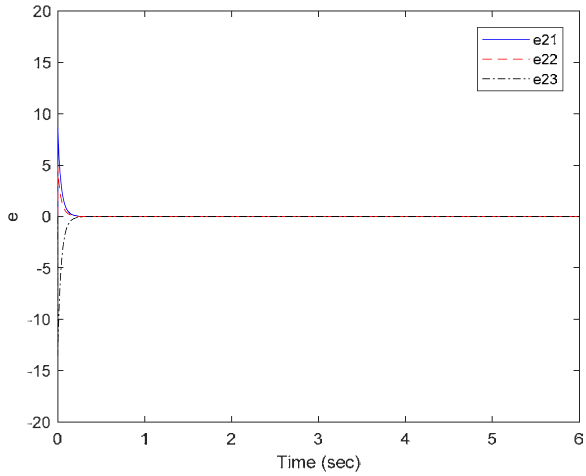

Figure 18. Synchronization error curve between two systems (27) and (28)

According to Figures 17 and 18, it is seen that errors have converged to zero fast. Table 1 compares the time taken for the error to reach 0.05.

Table 1. Comparing convergence time of integer order and fractional order chaotic systems

e23	e22	e21	e13	e12	e11	States/ Convergence time
0.313	0.256	0.288	0.344	0.307	0.300	Integer order
0.264	0.203	0.237	0.305	0.263	0.252	Fractional order

It can be seen from Table 1 that convergence time in fractional order mode is reduced compared to the integer order mode. It is seen in simulations that if time step of the integer order chaotic system is dt=0.01, the system tends to infinity and becomes unstable. This is also checked for the fractional order chaotic system and the result indicates that the system does not become unstable.

5. Conclusion

In this paper, a polynomial fuzzy control design strategy is presented for synchronization of multi-scroll fractional order and integer order Chen chaotic systems. Polynomial fuzzy systems can model a wider range of nonlinear systems. Fractional order systems can offer a more accurate model compared to integer order systems. Using SOS tools, stability of the system is investigated and feedback gains of the polynomial fuzzy controller are obtained. Simulations are performed for fractional order and integer order systems. Results show that the proposed scheme has synchronized two fractional order chaotic Chen systems using fuzzy control for two integer order and polynomial fractional order systems with a high speed. In future studies, intelligent optimization algorithms can be used to adjust parameters of the polynomial fuzzy controller such that synchronization error and convergence time are reduced.

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