Bubnov-Galerkin method for the Elastic Stress Analysis of Rectangular Plates under Uniaxial Parabolic Distributed Edge Loads

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***Abstract*-The classical two dimensional elasticity problem of a rectangular plate $(2a \times 2b)$ subjected to parabolically distributed edge loads applied at the two edges $x = \pm a$ was solved in this study using the Bubnov-Galerkin variational method. Stress formulation was adopted, and Airy stress potential function used to express the problem as a boundary value problem described using the non-homogeneous fourth order biharmonic equation in terms of the Airy's stress function. Airy's stress functions were assumed in terms of one and three unknown parameters, and coordinate shape functions that satisfied both the domain equations and the boundary conditions on the loaded edges. Bubnov-Galerkin variational equations were then solved to determine the unknown parameters, and hence the Airy's stress functions. The normal shear stress fields were then determined from the Airy's stress functions. The solutions obtained were found to satisfy all the stress boundary conditions along the edges $x = \pm a;\ y = \pm b$ as well as the domain equations. The Bubnov-Galerkin variational solutions were in agreement with solutions obtained by Timoshenko and Goodier, and by Nwoji et al.**

Keywords: *Bubnov-Galerkin variational method, Airy's stress potential function, biharmonic equation.*

I. INTRODUCTION

The general system of elasticity field equations consists of a system of fifteen equations in terms of fifteen unknowns: six stress components, six strain components and three displacement components [1, 2]. The system is very difficult to solve in closed analytical form, hence modified formulations have been developed; namely displacement formulation, and stress formulation. The displacement formulation is done by eliminating the stresses and strains from the general system of equations; thus yielding a system of three equations in terms of the three unknown displacement components. The stress formulation is done by eliminating the displacements and strains from the general system of governing equations. This generates a system of six equations in terms of the six unknown stress components.

Simplifications of the general elasticity problem in three dimensions is also achieved by seeking simplifications with regard to the distributions of stresses or strains. This yields plane (or two dimensional) elasticity problems of plane strain or plane stress type and axisymmetric elasticity problems of plane or space. The two dimensional elasticity problem of rectangular plates subjected to non-uniformly distributed edge loads are very common in engineering applications as components of aircraft panels, space craft panels, and machine panels. Accurate determination of the elastic stress distribution in such plates is very vital for the elastic design of such structures. Due to the complex nature of such elasticity problems, no mathematically exact solution has been obtained so far for thin rectangular plates subjected to non uniformly distributed edge loads [3].

Tang and Wang [3] solved the problem in an approximate way using Ritz method. They adopted Chebyshev polynomials as the stress function which satisfy the boundary conditions and then proceeded to apply the Ritz method to determine the distribution of in plane stresses of rectangular plates under non uniformly distributed edge loads based on the theory of elasticity principles. They studied rectangular plates under uniaxial and biaxial parabolic edge loads with the aid of the mathematical computational software Mathematica [3]. Their solutions satisfy the stress boundary conditions and agree with solutions obtained using the numerical tools of finite element method and differential quadrature method [3]. Nwoji et al [4] presented a variational Ritz method for solving the elastic stress analysis of rectangular plates $(2a \times 2b)$ under parabolically distributed edge loads applied at the two faces $x = \pm a.$ Their variational formulation applied energy principles and assumed the plate is in plane stress state. The Ritz method was then used, in their work, to obtain the first variation of the total energy functional which represents the equilibrium state of the plate under the distributed load [4]. They obtained solutions for the normal and shear stress fields for one unknown term and three unknown terms in the stress potential functions [4].

Plane elasticity problems have also been formulated and solved using stress functions. The stress function formulation is based on the general idea of developing a representation for the stress fields in the elastic body that satisfies the differential equations of equilibrium and yields a single governing equation from the compatibility statement [5, 6, 7].

Stress functions are scalar or vector potential functions that satisfy the differential equations of equilibrium as well as the compatibility equations (conditions); and from which the stress fields can be derived. The stresses are derivable from stress functions by taking partial derivatives of the potentials with respect to the spatial coordinate variables [8]. Airy stress functions are the most common stress functions formulated for plane elasticity problems. Airy stress function method reduces the general formulation of plane elasticity problems to a single governing equation in terms of a single scalar function. The resulting governing equation can then be solved using methods and techniques of applied mathematics to generate analytical or closed form solutions and numerical or approximate solutions.

The plane elasticity problem of rectangular plates consist of solving the fourth order biharmonic equation in terms of the Airy's stress functions, and finding the stresses from the Airy's stress function [9].

II. RESEARCH AIM AND OBJECTIVES

The general aim and objective of this study is to apply the Galerkin method to solve the elasticity problem of finding stresses in rectangular plates under inplane loads distributed parabolically on the two faces $x = \pm a$ of the plate. The specific objectives are:

(i) to formulate the problem as a boundary value problem in terms of Airy's stress potential functions

(ii) to formulate the Galerkin variational integral for the governing partial differential equation of the problem

(iii) to solve the Galerkin variational integral obtained for a one term solution and for a three term solution.

(iv) to obtain the stress fields from the Airy's stress potential functions

III. THEORETICAL FRAMEWORK

For the plane problems of elasticity, the stress compatibility equation is:

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = -\alpha\left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y}\right) \quad (1)$$

where $\alpha = \frac{1}{1-\mu}$ for plane stress

$\alpha = 1 + \mu$ for plane stress

: = Poisson's ratio, and F_x and F_y are body forces.
In the absence of body forces, the equations, simplify to:

$$\nabla^2(\sigma_{xx} + \sigma_{yy}) = 0 \quad (2)$$

$$\frac{\partial^2\sigma_{xx}}{\partial y^2} + \frac{\partial^2\sigma_{xx}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial x^2} + \frac{\partial^2\sigma_{yy}}{\partial y^2} = 0 \quad (3)$$

For the problem of rectangular plates under uniaxial tensile load distributed on the faces $x = \pm a$ according to the parabolic form:

$$\sigma_{xx}(x = \pm a,\ y) = p\left(1 - \frac{y^2}{b^2}\right) \quad (4)$$

and using the Airy's stress function $\phi(x, y)$ defined as:

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} \qquad \sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} \quad (5)$$

$$\tau_{xy} = -\frac{\partial^2\phi}{\partial x\partial y}$$

The governing equations become:

$$\frac{\partial^4\phi}{\partial x^4}+\frac{2\partial^4\phi}{\partial x^2\partial y^2}+\frac{\partial^4\phi}{\partial y^4}-\frac{2p}{b^2}=0 \tag{6}$$

$$\nabla^4\phi(x,y)-\frac{2p}{b^2}=0 \tag{7}$$

or

$$\nabla^2\nabla^2\phi(x,y)-\frac{2p}{b^2}=0 \tag{8}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}$

$$\nabla^4 = \nabla^2\nabla^2 = \frac{\partial^4}{\partial x^4}+\frac{2\partial^4}{\partial x^2\partial y^2}+\frac{\partial^4}{\partial y^4}$$

∇^2 is the two dimensional Laplacian operator and ∇^4 is the biharmonic differential operator.

IV. Application of Galerkin Method to a Plate Under Uniaxial Parabolic Inplane Load

The differential equation of equilibrium for the elastic problem of a rectangular plate subject to uniaxial inplane tensile load in the *xx* direction given by Equation (4), is given by:

$$\nabla^4\phi(x,y)=\frac{2p}{b^2} \tag{9}$$

on $|x|<a; \quad |y|<b$

subject to the boundary conditions

$$\begin{aligned}
&\frac{\partial^2\phi}{\partial x\partial y}=0 \quad \text{on} \quad |x|=a\\
&\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{on} \quad |x|=a\\
&\frac{\partial^2\phi}{\partial x\partial y}=0 \quad \text{on} \quad |y|=b\\
&\frac{\partial^2\phi}{\partial x^2}=0 \quad \text{on} \quad |y|=b
\end{aligned} \tag{10}$$

where $\phi(x,y)$ is the Airy's stress potential function

The governing partial differential equation can be expressed as Equation (11)

$$\nabla^2\phi(x,y)-\frac{2p}{b^2}=0=L\phi(x,y) \tag{11}$$

where $L(\phi)=\nabla^4\phi(x,y)-\frac{2p}{b^2}$ (12)

and L is a differential operator.

Shape functions for a rectangular plate $2a\times 2b$ with origin of coordinates at the center can be constructed from $(x^2-a^2)^2(y^2-b^2)^2$ since such functions satisfy the boundary conditions on the faces $|x|=a,$ and $|y|=b,$ and also qualify as a suitable Airy's stress potential function. Hence suitable shape functions for $\phi(x,y)$ are

$$f(x,y)=(x^2-a)^2(y^2-b^2)^2 \tag{13}$$

$$f_2(x,y)=x^2(x^2-a)^2(y^2-b^2)^2 \tag{14}$$

$$f_3(x,y)=y^2(x^2-a)^2(y^2-b^2)^2 \tag{15}$$

The Galerkin variational integral becomes

$$\iint_{R_{xy}} \left(\nabla^4 \phi - \frac{2p}{b^2} \right) \phi_i \, dxdy = 0 \tag{16}$$

where R_{xy} is the two dimensional domain of the plate given by

$$-a \le x \le a; \quad -b \le y \le b$$

For a one parameter choice of the Airy's stress function,

$$\phi_1(x, y) = c_1 (x^2 - a^2)^2 (y^2 - b^2)^2 \tag{17}$$

$$\phi_1(x, y) = c_1 f_1(x) g_1(y) \tag{18}$$

The Galerkin variational integral becomes:

$$\int_{-a}^{a}\int_{-b}^{b} \left(\nabla^4 \phi_1 - \frac{2p}{b^2} \right) f_1(x) g_1(y) dxdy = 0 \tag{19}$$

$$\int_{-a}^{a}\int_{-b}^{b} \left(\nabla^4 c_1 f_1(x) g_1(y) - \frac{2p}{b^2} \right) f_1(x) g_1(y) dxdy = 0 \tag{20}$$

$$c_1 \int_{-a}^{a}\int_{-b}^{b} \left(\nabla^4 f_1 g_1 \right) f_1 g_1 dxdy = \int_{-a}^{a}\int_{-b}^{b} \frac{2p}{b^2} f_1 g_1 dxdy \tag{21}$$

$$c_1 \int_{-a}^{a}\int_{-b}^{b} \left(\frac{\partial^4 f_1 g_1}{\partial x^4} + 2\frac{\partial^4 f_1 g_1}{\partial x^2 \partial y^2} + \frac{\partial^4 f_1 g_1}{\partial y^4} \right) f_1 g_1 dxdy = \frac{2p}{b^2} \int_{-a}^{a}\int_{-b}^{b} f_1 g_1 dxdy \tag{22}$$

$$c_1 \left\{ \int_{-b}^{b}\int_{-a}^{a} \frac{\partial^4 f_1 g_1}{\partial x^4} f_1 g_1 dxdy + 2\int_{-b}^{b}\int_{-a}^{a} \frac{\partial^4 f_1 g_1}{\partial x^2 \partial y^2} f_1 g_1 dxdy + \int_{-b}^{b}\int_{-a}^{a} \frac{\partial^4 f_1 g_1}{\partial y^4} f_1 g_1 dxdy \right\} = \frac{2p}{b^2} \int_{-b}^{b}\int_{-a}^{a} f_1 g_1 dxdy \tag{23}$$

$$c_1 \left\{ \int_{-b}^{b}\int_{-a}^{a} f_1^{iv} \, f_1 g_1^2 dxdy + 2\int_{-b}^{b}\int_{-a}^{a} f_1'' f_1 g_1'' g_1 dxdy + \int_{-b}^{b}\int_{-a}^{a} f_1^2 g_1^{iv} g_1 dxdy \right\} = \frac{2p}{b^2} \int_{-b}^{b}\int_{-a}^{a} f_1 g_1 dxdy \tag{24}$$

$$\nabla^4 \phi_1 = c_1 \left\{ 24(y^2 - b^2)^2 + 2(4(3x^2 - a^2)4(3y^2 - b^2) + 24(x^2 - a^2)^2 \right\} \tag{25}$$

$$\nabla^4 \phi_1 = c_1 \left\{ 24(y^2 - b^2)^2 + 32(3x^2 - a^2)(3y^2 - b^2) + 24(x^2 - a^2)^2 \right\} \tag{26}$$

$$c_1 \int_{-b}^{b}\int_{-a}^{a} \left[24(y^2 - b^2)^2 + 32(3x^2 - a^2)(3y^2 - b^2) + 24(x^2 - a^2)^2 \right] (y^2 - b^2) dxdy$$

$$= \frac{2p}{b^2} \int_{-b}^{b}\int_{-a}^{a} (x^2 - a^2)^2 (y^2 - b^2)^2 dxdy \tag{27}$$

Simplifying,

$$c_1 \int_{-b}^{b}\int_{-a}^{a} \Big[24(y^2 - b^2)^4 (x^2 - a^2)^2 + 32(3x^2 - a^2)(x^2 - a^2)^2 (3y^2 - b^2)(y^2 - b^2)^2$$

$$+ 24(x^2 - a^2)^4 (y^2 - b^2)^2 \Big] dxdy = \frac{2p}{b^2} \int_{-b}^{b}\int_{-a}^{a} (x^2 - a^2)^2 (y^2 - b^2)^2 dxdy \tag{28}$$

$$c_1 \left\{ I_1 + I_2 + I_3 \right\} = \frac{2p}{b^2} I_4 \tag{29}$$

where

$$I_1 = 24 \iint (y^2 - b^2)^4 (x^2 - a^2)^2 dxdy$$

$$I_1 = 24 \int_{-b}^{b} (y^2 - b^2)^4 dy \int_{-a}^{a} (x^2 - a^2)^2 dx \tag{30}$$

$$I_2 = 32\int_{-a}^{a}(3x^2 - a^2)(x^2 - a^2)^2\,dx \int_{-b}^{b}(3y^2 - b^2)(y^2 - b^2)^2\,dy \tag{31}$$

$$I_3 = 24\int_{-a}^{a}(x^2 - a^2)^4\,dx \int_{-b}^{b}(y^2 - b^2)^2\,dy \tag{32}$$

$$I_4 = \int_{-a}^{a}(x^2 - a^2)^2\,dx \int_{-b}^{b}(y^2 - b^2)^2\,dy \tag{33}$$

Using online Wolfram integration software, we obtain:

$$\int_{-a}^{a}(x^2 - a^2)^4\,dx = \frac{256}{315}a^9 \tag{34}$$

$$\int_{-a}^{a}(x^2 - a^2)^2\,dx = \frac{16}{15}a^5 \tag{35}$$

$$\int_{-b}^{b}(y^2 - b^2)^4\,dy = \frac{256}{315}b^9 \tag{36}$$

$$\int_{-b}^{b}(y^2 - b^2)^2\,dy = \frac{16}{15}b^5 \tag{37}$$

$$\int_{-a}^{a}(3x^2 - a^2)(x^2 - a^2)^2\,dx = -0.609523809a^7 \tag{38}$$

$$\int_{-b}^{b}(3x^2 - b^2)(y^2 - b^2)^2\,dy = -0.609523809b^7 \tag{39}$$

Thus,

$$I_1 = 24\cdot\frac{256}{315}b^9\cdot\frac{16}{15}a^5 \tag{40}$$

$$I_2 = 32(-0.609523809a^7)(-0.609523809b^7)$$

$$I_2 = 11.88861676a^7b^7 \tag{41}$$

$$I_3 = 24\cdot\frac{256}{315}a^9\cdot\frac{16}{15}b^5 \tag{42}$$

$$I_4 = \frac{16}{15}a^5\cdot\frac{16}{15}b^5 \tag{43}$$

By substitution into Equation (29), we obtain:

$$c_1\left\{24\cdot\frac{256}{315}\cdot\frac{16}{15}b^9a^5 + 11.88861676a^7b^7 + 24\cdot\frac{256}{315}\cdot\frac{16}{15}a^9b^5\right\} = \frac{2p}{b^2}\cdot\frac{16}{15}\cdot\frac{16}{15}a^5b^5 \tag{44}$$

$$c_1\left\{\frac{98304}{4725}(b^9a^5 + a^9b^5) + 11.88861676a^7b^7\right\} = \frac{512}{225}pa^5b^3 \tag{45}$$

$$c_1\left(20.80507937(a^5b^9 + a^9b^5) + 11.88861676a^7b^7\right) = \frac{512}{225}pa^5b^3 \tag{46}$$

$$c_1 = \frac{2.27555pa^5b^3}{\left[20.80507937(a^5b^9 + a^9b^5) + 11.88861676a^7b^7\right]} \tag{47}$$

$$c_1 = \frac{pa^{-6}\alpha^{-2}}{\frac{64}{7}(1+\alpha^4) + \frac{256}{49}\alpha^2} \tag{48}$$

where $\alpha = b/a$; α is the plate aspect ratio.

$$c_1 = \alpha^{-2}F(\alpha)pa^{-6} = F_1(\alpha)pa^{-6} \tag{49}$$

where, $F(\alpha) = \left[\frac{64}{7}(1+\alpha^4) + \frac{256}{49}\alpha^2\right]^{-1}$

Then,

$$\phi(x,y) = \phi_0(x,y) + \phi_1(x,y) \tag{50}$$

$$\phi(x,y) = \frac{py^2}{2}\left(1 - \frac{y^2}{6b^2}\right) + F_1(\alpha)pa^{-6}f_1g_1 \tag{51}$$

$$\phi(x,y) = \frac{py^2}{2}\left(1 - \frac{y^2}{6b^2}\right) + F_1(\alpha)pa^{-6}(x^2-a^2)^2(y^2-b^2)^2\frac{1}{2} \tag{52}$$

The stresses are obtained from the Airy's stress potential function as:

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = -4F_1(\alpha)\left(1-\frac{3x^2}{a^2}\right)\left(1-\frac{y^2}{b^2}\right)^2 p\left(\frac{b^4}{a^4}\right) \tag{53}$$

$$\sigma_{yy}(x=0,y) = -4F_1(\alpha)\left(1-\frac{y^2}{b^2}\right)^2 p\alpha^4 \tag{54}$$

$$\sigma_{yy}(x,y=0) = -4F_1(\alpha)\left(1-\frac{3x^2}{a^2}\right)p\alpha^4 \tag{55}$$

$$\sigma_{yy}(x=0,y=0) = -4F_1(\alpha)p\alpha^4 \tag{56}$$

$$\tau_{xy} = \frac{-\partial^2\phi}{\partial x\partial y} = -16F_1(\alpha)\left(\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^2}-1\right)xyp\alpha^2a^{-2} \tag{57}$$

$$\tau_{xy}(0,0) = 0 \tag{58}$$

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2} = p\left(1-\frac{y^2}{b^2}\right) + 4F_1(\alpha)pa^{-6}(3y^2-b^2)(x^2-a^2)^2 \tag{59}$$

For square plates, $\alpha = 1$, $a = b$

$$c_1 = 0.042535pa^{-6} \tag{60}$$

$$\sigma_{yy} = -0.1702p\left(1-\frac{3x^2}{a^2}\right)\left(1-\frac{y^2}{a^2}\right) \tag{61}$$

$$\tau_{xy} = -0.68056\left(1-\frac{y^2}{a^2}\right)\left(1-\frac{x^2}{a^2}\right)\frac{xy}{a^2} \tag{62}$$

$$\sigma_{xx} = p\left(1-\frac{y^2}{b^2}\right) - 0.1702p\left(1-\frac{x^2}{a^2}\right)^2\left(1-\frac{3y^2}{a^2}\right) \tag{63}$$

For a three term Galerkin solution, we have:

$$\phi_3 = c_1(x^2-a^2)^2(y^2-b^2)^2 + c_2x^2(x^2-a^2)^2(y^2-b^2)^2 + c_3(x^2-a^2)^2y^2(y^2-b^2)^2 \tag{64}$$

$$\phi_3 = c_1f_1g_1 + c_2f_2g_1 + c_3f_1g_2 \tag{65}$$

$$\nabla^4\phi_3 = \nabla^4(c_1f_1g_1 + c_2f_2g_1 + c_3f_1g_2) \tag{66}$$

$$\nabla^4\phi_3 = c_1\nabla^4f_1g_1 + c_2\nabla^4f_2g_1 + c_3\nabla^4f_1g_2 \tag{67}$$

The Galerkin variational integrals are:

$$\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\phi_3 - \frac{2p}{b^2}\right)f_1g_1\,dxdy = 0 \tag{68}$$

$$\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\phi_3-\frac{2p}{b^2}\right)f_2g_1dxdy=0 \tag{69}$$

$$\int_{-b}^{b}\int_{-a}^{a}\left(\nabla^4\phi_3-\frac{2p}{b^2}\right)f_1g_2dxdy=0 \tag{70}$$

or,

$$\iint\left\{\left[c_1\nabla^4 f_1g_1+c_2\nabla^4 f_2g_1+c_3\nabla^4 f_1g_2\right]-\frac{2p}{b^2}\right\}f_1g_1dxdy=0 \tag{71}$$

$$\iint\left\{\left(c_1\nabla^4 f_1g_1+c_2\nabla^4 f_2g_1+c_3\nabla^4 f_1g_2\right)-\frac{2p}{b^2}\right\}f_2g_1dxdy=0 \tag{72}$$

$$\iint\left\{\left(c_1\nabla^4 f_1g_1+c_2\nabla^4 f_2g_1+c_3\nabla^4 f_1g_2\right)-\frac{2p}{b^2}\right\}f_1g_2dxdy=0 \tag{73}$$

Evaluating the integrals and simplifying,

$$c_1\left(\frac{64}{7}+\frac{256}{49}\frac{b^2}{a^2}+\frac{64}{7}\frac{b^4}{a^4}\right)+c_2a^2\left(\frac{64}{77}+\frac{64}{49}\frac{b^4}{a^4}\right)+c_3a^2\left(\frac{64}{49}\frac{b^2}{a^2}+\frac{64}{77}\frac{b^6}{a^6}\right)=\frac{p}{a^4b^2} \tag{74}$$

$$c_1\left(\frac{64}{11}+\frac{64}{7}\frac{b^4}{a^4}\right)+c_2a^2\left(\frac{192}{143}+\frac{256}{77}\frac{b^2}{a^2}+\frac{192}{7}\frac{b^4}{a^4}\right)+c_3a^2\left(\frac{64}{77}\frac{b^2}{a^2}+\frac{64}{77}\frac{b^6}{a^6}\right)=\frac{p}{a^4b^2} \tag{75}$$

$$c_1\left(\frac{64}{7}+\frac{64}{11}\frac{b^4}{a^4}\right)+c_2a^2\left(\frac{64}{77}+\frac{64}{77}\frac{b^4}{a^4}\right)+c_3a^2\left(\frac{192}{7}\frac{b^2}{a^2}+\frac{256}{77}\frac{b^4}{a^4}+\frac{192}{143}\frac{b^6}{a^6}\right)=\frac{p}{a^4b^2} \tag{76}$$

Let $b/a=\alpha$

$$c_1\left(\frac{64}{7}(1+\alpha^4)+\frac{256}{49}\alpha^2\right)+c_2a^2\frac{64}{7}\left(\frac{1}{11}+\frac{\alpha^4}{7}\right)+c_3a^2\frac{64}{7}\left(\frac{\alpha^2}{7}+\frac{\alpha^6}{11}\right)=\frac{p}{a^4b^2} \tag{77}$$

$$c_1 64\left(\frac{1}{11}+\frac{\alpha^4}{7}\right)+c_2a^2\left(\frac{192}{143}+\frac{256}{77}\alpha^2+\frac{192}{7}\alpha^4\right)+c_3a^2\frac{64}{77}\left(\alpha^2+\alpha^6\right)=\frac{p}{a^4b^2} \tag{78}$$

$$c_1 64\left(\frac{1}{7}+\frac{\alpha^4}{11}\right)+c_2a^2\frac{64}{77}\left(1+\alpha^4\right)+c_3a^2\left(\frac{192}{7}\alpha^2+\frac{256}{77}\alpha^4+\frac{192}{143}\alpha^6\right)=\frac{p}{a^4b^2} \tag{79}$$

For square plates, $a=b$, $\alpha=1$, and we obtain:

$$23.5102c_1+2.1373c_2a^2+2.1373c_3a^2=\frac{p}{a^6} \tag{80}$$

$$14.9610c_1+32.0959c_2a^2+1.6623c_3a^2=\frac{p}{a^6} \tag{81}$$

$$14.9610c_1+1.6623c_2a^2+32.0505c_3a^2=\frac{p}{a^6} \tag{82}$$

In matrix form,

$$\begin{pmatrix}23.5102 & 2.1373a^2 & 2.1373a^2\\ 14.9610 & 32.0959a^2 & 1.6623a^2\\ 14.9610 & 1.6623a^2 & 32.0505a^2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\\ c_3\end{pmatrix}=\begin{pmatrix}pa^{-6}\\ pa^{-6}\\ pa^{-6}\end{pmatrix} \tag{83}$$

Solving, $c_1=0.04040pa^{-6}$ (84)

$$c_2=c_3=0.01174pa^{-8} \tag{85}$$

Thus, the Airy's stress function for a three term solution for a square plate is:

$$\phi(x,y) = \frac{py^2}{2}\left(1-\frac{y^2}{6b^2}\right) + 0.04040\,pa^{-6}(x^2-a^2)^2(y^2-b^2)^2$$
$$+\ 0.01174\,pa^{-8}(x^2+y^2)(x^2-a^2)^2(y^2-b^2)^2 \quad (86)$$

The stresses are then obtained from $\phi(x,y)$ as:

$$\sigma_{xx} = \frac{\partial^2\phi}{\partial y^2}$$

$$\sigma_{xx} = p\left(1-\frac{y^2}{b^2}\right) + 0.04040\,pa^{-6}(x^2-a^2)^2\cdot 4(3y^2-b^2)$$
$$+\ 0.01174\,pa^{-8}(x^2-a^2)^2(2x^2-4b^2x^2+30y^4-24b^2y^2+2b^4) \quad (87)$$

$$\sigma_{xx} = p\left(1-\frac{y^2}{b^2}\right) + 0.1616\,pa^{-6}(x^2-a^2)^2(3y^2-b^2)$$
$$+\ 0.01174\,pa^{-8}(x^2-a^2)^2(2x^2-4b^2x^2+30y^4-24b^2y^2+2b^4) \quad (88)$$

$$\sigma_{xx}(x=0) = p\left(1-\frac{y^2}{b^2}\right) + 0.1616\,pa^{-6}(3y^2-b^2)\ +\ 0.01174\,pa^{-4}(30y^4-24b^2y^2+2b^4) \quad (89)$$

$$\sigma_{yy} = \frac{\partial^2\phi}{\partial x^2} = 0.1616\,pa^{-6}(3x^2-a^2)\ +\ 0.01174\,pa^{-8}(y^2-b^2)^2(2y^2-4a^2x^2+30x^4-24a^2x^2+2a^4) \quad (90)$$

$$\sigma_{yy}(x=0) = -0.1616\,p + 0.01174\,pa^{-8}(y^2-b^2)^2(2y^2+2a^4) \quad (91)$$

$$\sigma_{yy}(0,0) = -0.13812\,p \quad (92)$$

Thus, the distribution of normal stress σ_{xx} on the cross-sectional plane x = 0 of a square plate under the load σ_{xx} at $|x| = a$ is given by

$$\sigma_{xx}(x=0,y) = p\left(1-\frac{y^2}{b^2}\right) - 0.1616\,p\left(1-\frac{3y^2}{a^2}\right) + 0.0235\,p\left(1-\frac{12y^2}{a^2}+\frac{15y^4}{a^4}\right) \quad (93)$$

The variation of the normal stress field σ_{xx} on the cross-sectional plane x = 0 for square plate under uniaxial parabolic load $\sigma_{xx} = p\left(1-\frac{y^2}{b^2}\right)$ applied at the faces $x = \pm a$ is shown displayed in Table 1 for one parameter Galerkin solution, and in Table 2 for three term Galerkin solution.

TABLE 1: One term (parameter) Galerkin solution for distribution of normal stress σ_{xx} on the plane x = 0 for square plates under parabolic load $\sigma_{xx} = p\left(1-\frac{y^2}{b^2}\right)$ on $x = \pm a$

y/a	$\sigma_{xx}(x=0) = p\left(1-\frac{y^2}{a^2}\right) - 0.1702\,p\left(1-\frac{3y^2}{a^2}\right)$
0	0.8298p
0.2	0.8102p
0.4	0.7515p
0.6	0.6536p
0.8	0.5166p
1.0	0.3404p

TABLE 2: Three term (parameter) Galerkin solution for normal stress distribution on the plane $x = 0$ for square plates under parabolic load

$$\sigma_{xx} = p\left(1-\frac{y^2}{b^2}\right) \text{ at } x = \pm a$$

y/a	$\sigma_{xx}(x=0)$
0	0.8619p
0.2	0.8306p
0.4	0.7434p
0.6	0.6206p
0.8	0.4961p
1.0	0.4172p

V. DISCUSSION OF RESULTS

The Bubnov-Galerkin method has been successfully used to determine the normal and 'shear stress distributions in a thin rectangular plate $2a \times 2b$ subjected to a parabolic distribution of loads on the two edges $x = \pm a$. The parabolic distribution of normal stress on the edges $x = \pm a$ was given as Equation (4) while the other edges were considered free of normal and shear stresses. The elastic stress analysis problem was formulated as a boundary value problem in terms of Airy's stress potential function and the governing partial differential equation found as Equation (6) or (7) or (8). Airy's stress potential functions were approximated as linear combinations of coordinate shape functions that satisfy both the stress boundary conditions as well as the domain conditions as Equations (13), (14) and (15). The Bubnov-Galerkin variational integral was found as Equation (20) for a one parameter Airy's stress function and Equations (68-70) for a three parameter Airy's stress function. Solution of the Bubnov-Galerkin variational integral Equation (20) for a one parameter Airy's stress function yielded the solution for the Airy's stress potential function as Equation (52). The normal and shear stress fields were then determined from the Airy's stress function as Equations (53), (57) and (59). The solutions were also presented for square plates as Equations (61-63) for a one parameter Bubnov-Galerkin solution.

The Airy's stress potential function considered for a three parameter solution was given as Equation (64). The Bubnov-Galerkin variational integral equations yielded a system of three equations in terms of the three unknown parameters of the Airy's stress function and were given as Equations (71) – (73). The equations were solved for square plates to obtain the three parameters as Equations (84) and (85); hence the three parameter Airy's stress potential function was found as Equation (86). The stresses were found as Equations (88) and (90). The normal stress distribution σ_{xx} over the cross-sectional plane $x = 0$ shows that as the plate aspect ratio increases, the normal stress distribution over the section $x = 0$ becomes more uniform. For instance, when $a/b = 2$, the Airy's stress constants for a three parameter Bubnov-Galerkin solution are $c_1 = 0.07983pa^{-4}b^{-2}$, $c_2 = 0.1250pa^{-6}b^{-2}$ and $c_3 = 0.01826pa^{-6}b^{-2}$ and the distribution of σ_{xx} over the cross-section $x = 0$, are given in Table 3 for $a/b = 2$, for various values of y/b yielding an average value of $\frac{2}{3}p$.

Table 3: Normal stress distribution over the cross-section $x = 0$ for $a/b = 2$

y/b	0	0.2	0.4	0.6	0.8	1.0
$\sigma_{xx}(x=0)$	0.690p	0.684p	0.669p	0.653p	0.649p	0.675p

The results of this study agree remarkably well with those presented in Timoshenko and Goodier [1].

VI. CONCLUSION

From the study the following conclusions can be made:

(i) The elasticity prolem of rectangular plates subject to uniaxial distributed edge loads is described by nonhomogeneous fourth order biharmonic equation in terms of the Airy's stress potential function in a stress potential function formulation.

(ii) The Bubnov-Galerkin variational method is an effective mathematical tool for the approximate solution of the determination of normal stresses and shear stress distributions in rectangular plates subjected to a parabolic distribution of edge loads at the face $x = \pm a$ in one direction.

(iii) A one parameter approximation of the Airy's stress potential function in the Bubnov-Galerkin variational equation yielded sufficiently accurate results for practical purposes.

(iv) A three term approximation of the Airy's stress potential functions in the Bubnov-Galerkin variational equation yielded more accurate results.

(v) As the plate aspect ratio increases, and the plate becomes very long in one direction relative to the other, the normal stress distribution over the cross-section of the plane $x = 0$ becomes uniform; a result that agrees with logical reasoning.

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