Simulation of an Inverse Heat Conduction Boundary Estimation Problem Based on State Space Model

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Abstract—This paper presents a non-iterative procedure based on state space model for estimation of boundary conditions in heat conduction problems. The conduction governing equations are represented in the form of state space model. The non-iterative nature of the solution procedure in this method makes the solution very fast. Varieties of time-varying heat fluxes are simulated to realize the performance of the method. The results show that the estimation is quite accurate even at reasonably high levels of measurement noise. The results also confirm that this method can estimate sudden changes in boundary conditions accurately.

Keyword-Inverse Heat Conduction, State Space, Conjugate-gradient Method, Boundary Estimation

I. INTRODUCTION

The concept of inverse problems has found application in almost all disciplines of science and technology in general and in heat transfer in particular. Although few of the tools used for solution of these problems are based on an analytical solution of the inverse problem, the majority of the techniques rely on numerical methods to solve inverse problems. The heat transfer processes such as jet impingement cooling, occurring in industrial applications requires thorough knowledge of thermal boundary conditions such as local surface temperature, surface heat flux or convection heat transfer coefficient. In practical situations, these unknown parameters and/or functions are to be determined from transient temperature measurements at one or more locations.

The problem of deriving an unknown boundary condition from a set of measured temperatures is known as inverse heat conduction problem (IHCP). On the other hand, the more common route of determining the temperature distribution in a body subject to known boundary conditions is often referred to as the direct problem. Inverse problems are treated as mathematically ill-posed for two reasons [1]. Firstly, they are often unstable, being extremely sensitive to small variations in input like random measurement errors. Secondly, though the existence of the solution of inverse problems can be intuititively argued from physical considerations, it can be formally proved only for a few cases.

Several approaches have been used for solving IHCP problems. The common approaches include variational methods like conjugate gradient methods [2], regularized deterministic methods like quasi-Newton method [1] and hybrid techniques of Laplace transform and finite difference method [3]. In addition, methods like genetic algorithm [4] and artificial neural networks [5] have also been applied for determination of heat transfer boundary conditions. Many IHCP algorithms have been derived from techniques used in control systems like applying state space models [6] and use of observers [6,7] for estimating non-measured temperatures.

The majority of IHCP algorithms are iterative in nature, which requires large computation times. Only a few approaches [8] use direct methods for estimation of thermal boundary conditions. The objective of the present work is to develop and demonstrate a non-iterative technique for estimation of thermal boundary condition using a state space representation of the governing energy equation.

II. THE CONDUCTION STATE SPACE MODEL

The proposed approach for determination of surface boundary conditions has been demonstrated here with a transient one-dimensional heat conduction problem subject to time-varying boundary conditions.

A. *Direct Problem*

The direct problem involves determination of temperature in a one-dimensional slab heated by a time-varying but prescribed heat flux on one side and insulated on the other, as shown in Fig. 1. The governing differential equation and the initial and boundary conditions are as follows:

$$\rho C \frac{\partial T}{\partial t} = \lambda \frac{\partial^2 T}{\partial x^2} \tag{1a}$$

with the boundary conditions

$$\frac{\partial T}{\partial x} = 0 \text{ at x=L} \tag{1b}$$

and $$-\frac{\partial T}{\partial x} = \frac{q(t)}{\lambda} \text{ at x=0} \tag{1c}$$

B. Inverse Problem

The inverse problem involves determination of the surface heat flux q(t) from the measured values of the temperature at x = L. The first step in the inverse problem is to express the problem in state space form by spatially discretising the governing differential equation as shown in Fig. 1(b). The discretised equations, after application of boundary conditions become

$$\frac{\dot{T}_1}{2} = \frac{\lambda}{\rho C(\Delta x)^2}\left[T_2 - T_1\right] + \frac{q}{\rho C \Delta x} \tag{2a}$$

$$\dot{T}_i = \frac{\lambda}{\rho C(\Delta x)^2}\left[T_{i+1} - 2T_i + T_{i-1}\right], \text{i = 2,…, N-1} \tag{2b}$$

$$\frac{\dot{T}_N}{2} = \frac{\lambda}{\rho C(\Delta x)^2}\left[-T_N + T_{N-1}\right] \tag{2c}$$

Expressed in matrix notations, the above equations become

$$\{\dot{\mathbf{T}}\} = [\mathbf{A}]\{\mathbf{T}\} + [\mathbf{B}]\{\mathbf{u}\} \tag{2d}$$

In the above equation, the matrices **A** and **B** and the input vector **u** are as follows:

$$\mathbf{A} = \frac{\lambda}{\rho C(\Delta x)^2}\begin{bmatrix} -2\ 2\,0\ 0...0 \\ 1\ \text{-}2\ 1\ 0...0 \\ ... \\ 0\ 0...1\ \text{-}2\ 1 \\ 0\ \ 0...2\ \text{-}2 \end{bmatrix};\ \mathbf{B} = \frac{1}{\rho C(\Delta x)}\begin{bmatrix} 1\ 0\ 0...0 \\ 0\ 0\ 0...0 \\ ... \\ 0\ 0\ 0...0 \\ 0\ 0\ 0...0 \end{bmatrix};\ \mathbf{u} = \begin{Bmatrix} q \\ 0 \\ ... \\ 0 \\ 0 \end{Bmatrix} \tag{2e}$$

Since sensors are placed at only P of the above N points, the state vector is partitioned into vectors of size P×1 and (N-P) ×1 representing measured states and non-measured states, which need to be estimated. Thus Eq. (2d) becomes

$$\begin{Bmatrix} \dot{\mathbf{T}}_\mathbf{m} \\ \dot{\mathbf{T}}_\mathbf{n} \end{Bmatrix} = [\mathbf{A}]\begin{Bmatrix} \mathbf{T}_\mathbf{m} \\ \mathbf{T}_n \end{Bmatrix} + [\mathbf{B}]\{\mathbf{u}\} \tag{3}$$

For a given time series of temperatures at the sensor locations, $\dot{\mathbf{T}}_\mathbf{m}$ can be easily obtained. Thus the problem reduces to determining the unknowns $\mathbf{T}_n$, $\dot{\mathbf{T}}_n$ and $\mathbf{u}$. As the problem evolves in time, using the available values of $\mathbf{T}_n$ estimated at earlier time steps, $\dot{\mathbf{T}}_n$ can be estimated. Thus starting from an initial estimate of $\dot{\mathbf{T}}_n$, presumably equal to zero, the following system of algebraic equations need to be solved.

$$[\mathbf{C}]\begin{Bmatrix} \mathbf{T}_\mathbf{n} \\ \mathbf{u} \end{Bmatrix} = \begin{Bmatrix} \dot{\mathbf{T}}_m \\ \dot{\mathbf{T}}_\mathbf{n} \end{Bmatrix} + [\mathbf{D}]\{\mathbf{T}_m\} \tag{4}$$

The coefficient matrices, **C** and **D** are obtained from algebraic manipulation of Eq. (2). This approach has earlier been successfully implemented in the context of feedback tracking control by Ghosh et al. [9] and Mishra et al. [10]. As is evident from Eq. (4), this method requires as many sensors as the number of boundary conditions to be determined.

III. RESULTS AND DISCUSSION

The method is illustrated with the example of transient heating of a one-dimensional slab under time-varying heat flux. In the simulated experiments, first the temperature in the plate is determined for a known heat flux by solving the direct problem. The temperature data at the sensor location, to be used in the inverse problem, is obtained from this solution. To simulate the effect of measurement noise in actual experiments, random errors have been added to the temperatures obtained from the direct simulation. Thus the temperature data that are used as inputs for the inverse problem are generated as:

$$Y(t) = T_n(t) + \sigma e(t) \tag{5}$$

where $0 \le e(t) \le 1$ is a random number and σ represents the amplitude of the noise. The thermo-physical properties used in the simulation correspond to that of steel and the values used are ρ= 7850 kg/m^3, C = 500 J/(kg K) and λ = 54 W/mK. Two time-varying heat fluxes are considered in this work, a step pulse and a triangular pulse. The step pulse represents a challenging condition for any inverse estimation technique due to the instantaneous jump in the boundary value. The triangular heat flux, on the other hand, represents a condition where the boundary heat flux changes at a given rate.

A. Step Pulse

In Fig. 2 (a) and (b), we investigate the effects of time step sizes and pulse durations on estimation of step heat flux. In Fig. 2 (a), the total pulse duration is 10s in all simulations. Simulation has been performed for three different time step sizes (i.e., Δt = 0.1s, 0.05s and 0.01s) in descending order. At all the time step sizes, the time-varying heat fluxes estimated by the inverse solutions show excellent agreement with the exact heat flux except for the initial part of the step. It can be seen that, the higher the value of time step size the higher is the delay before which the estimated boundary accurately matches the exact one. In Fig. 2 (b), we show the characteristics of estimated heat fluxes from the inverse solutions with the variations of total pulse durations. Here, for the convenience of presentation, we represent the time, non-dimensionalised with the total time. Simulation has been carried out for three different total pulse durations (i.e., $t_{\max}$ = 10s, 50s, and 100s) and a simulation time step of 1s. It is observed, again, that the estimated heat flux agrees very well with the exact one though the agreement is better for larger pulse durations. This is because of better temporal resolution obtained with the fixed time step.

In Fig. 3, we compare the estimation of step heat flux using the present State Space Method (SSM) with Conjugate Gradient Method (CGM), which is a widely used method, for different measurement noise levels (i.e., σ = 0.0, 0.02, 0.1, and 1.0). The algorithm for the Conjugate Gradient Method is adopted from Ozisik [11]. The observation time is 10.0s and the time step size Δt is 0.1s in all the numerical simulations. The computation time for the CGM is orders of magnitude higher than the present method, which is a non-iterative procedure. Both the simulation time and temporal resolutions are representative of actual experiments. At all noise levels, the estimation using SSM can follow the step change much better than the CGM. For the estimation using the CGM, the response to the step change is rather sluggish. At zero and low noise levels, the estimation using SSM is very accurate. As the noise level increases, the SSM estimation still follows the exact solution fairly accurately but the estimation becomes increasingly noisy showing fluctuations about the exact solution. A comparison of the results using the two methods show that up to a noise level of σ = 0.1, SSM estimates the heat flux better. But at high noise levels (σ = 0.1), both the methods give noisy output but the SSM prediction has a higher noise level. However, actual noise levels in the experiment are expected to be lower.

B. Triangular Pulse

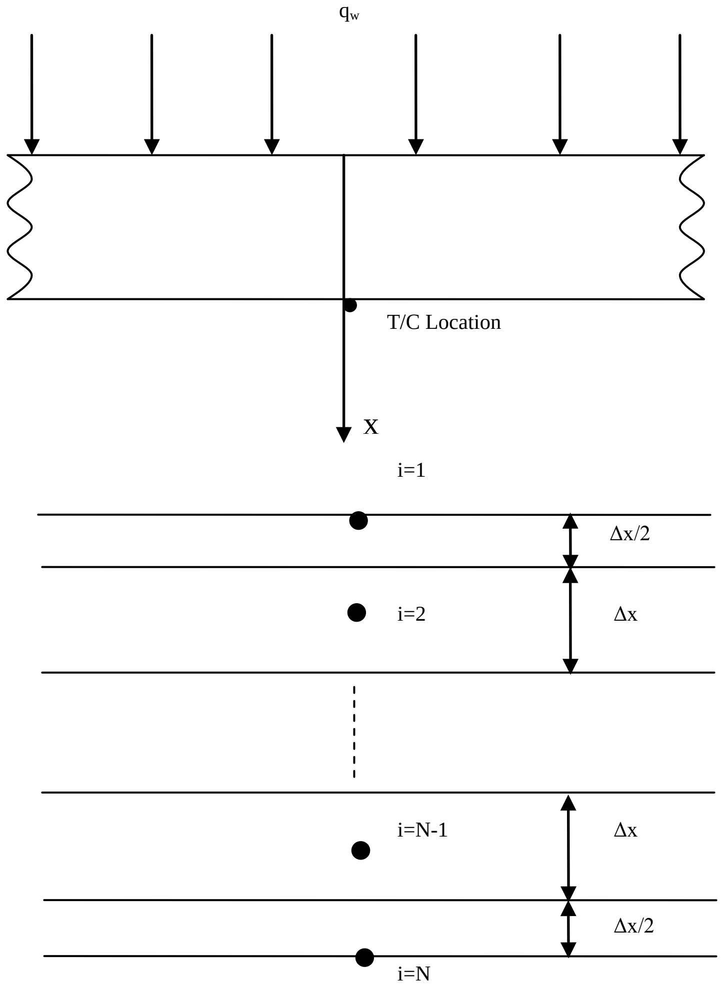
In Fig. 4(a) and (b), we investigate the effects of time step sizes and pulse durations on estimation of triangular heat flux. In Fig. 4 (a), we present the variation of triangular heat fluxes estimated from the inverse solutions for a total pulse duration of 10 s. Simulation has been performed for three different time step sizes (i.e., Δt = 0.05s, 0.01s and 0.1s). It can be seen that, the results are much less sensitive to time step size for triangular pulse than for step pulse. This is expected as the variation in heat flux is more gradual in this case. In Fig. 4(b), we represent the characteristics of estimated triangular heat fluxes from the inverse solutions with the variations of total pulse durations. For this configuration, the variation in pulse duration for a given rate of ramping signifies different peak heat flux also. Simulation has been carried out for three different total pulse durations (i.e., $t_{\max}$ = 10s, 20s, and 50s) and simulation time of 1s. Excellent agreement is observed between the exact and the estimated heat fluxes in all the cases.

In Fig. 5, we compare the estimation of triangular heat flux using State Space Model (SSM) and Conjugate Gradient Model (CGM) for different measurement noise levels (i.e., σ = 0.0, 0.02, 0.1, and 1.0). The observation time is 10.0s and the time step size Δt is 0.1s in all numerical simulations. It can be seen that the time-varying heat fluxes estimated by the inverse solutions agree very well with the exact heat fluxes at all noise levels for both the methods. The sensitivity of the estimation to measurement noise is significantly lower for triangular pulse compared to step pulse.

From the above simulation results it can be predicted that, the proposed methods are effective enough in tracking the unknown time varying heat fluxes in one dimensional heat conduction problems.

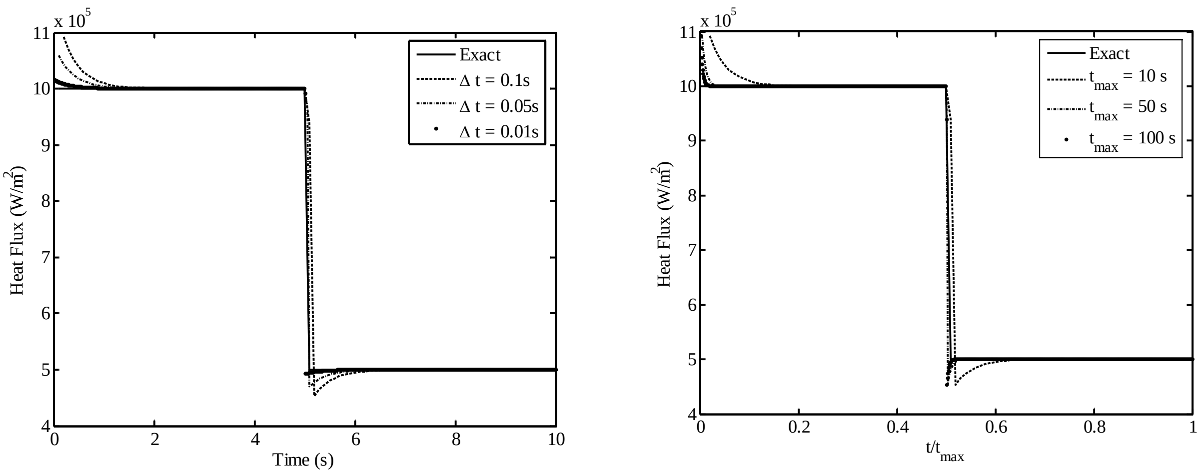
C. Figures

(a)



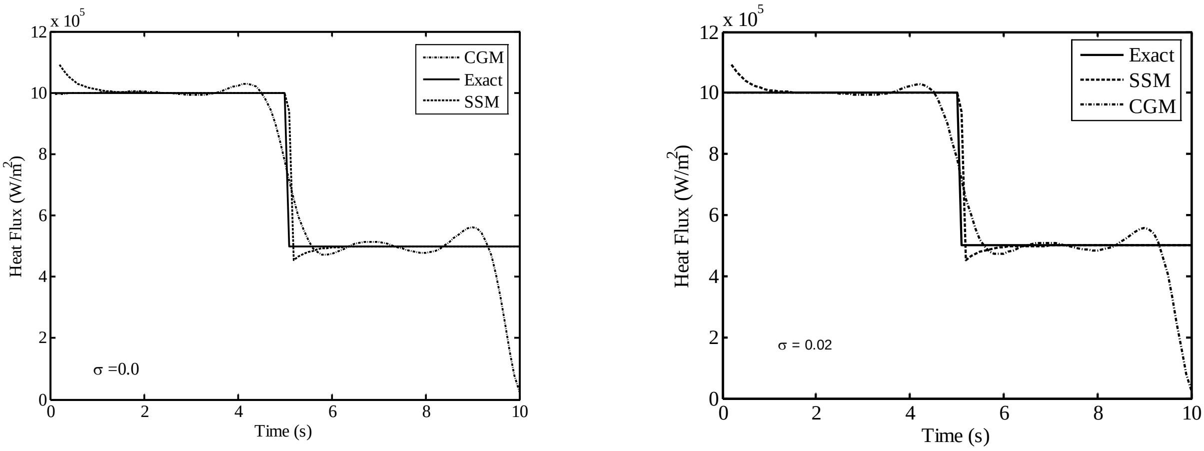
(b)

Fig. 1. (a) Schematic of the geometry and (b) discretisation scheme



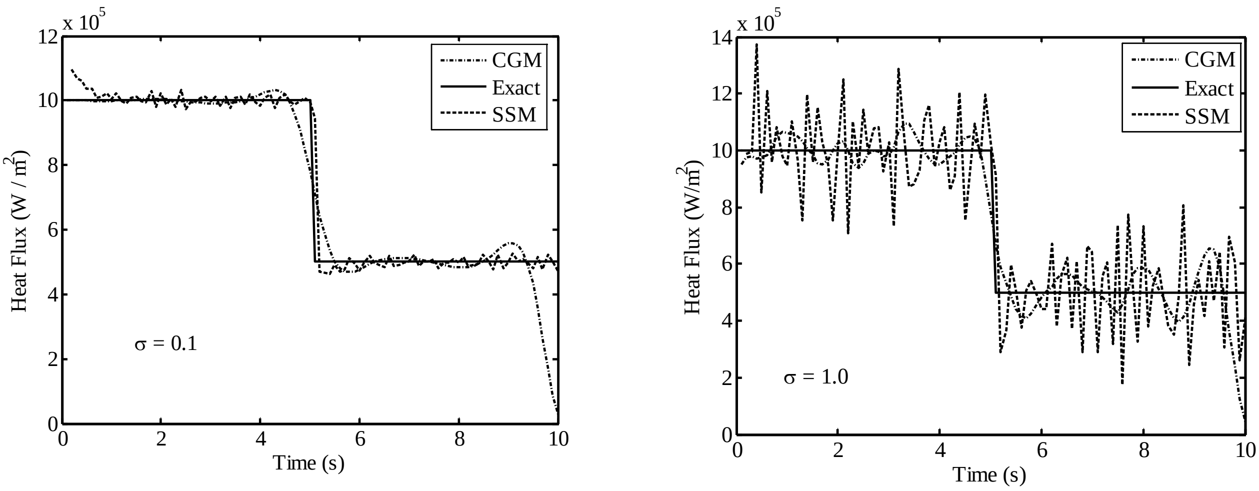
(a) (b)

Fig. 2. Effect of (a) time step sizeand (b) pulse duration on estimation of step heat flux



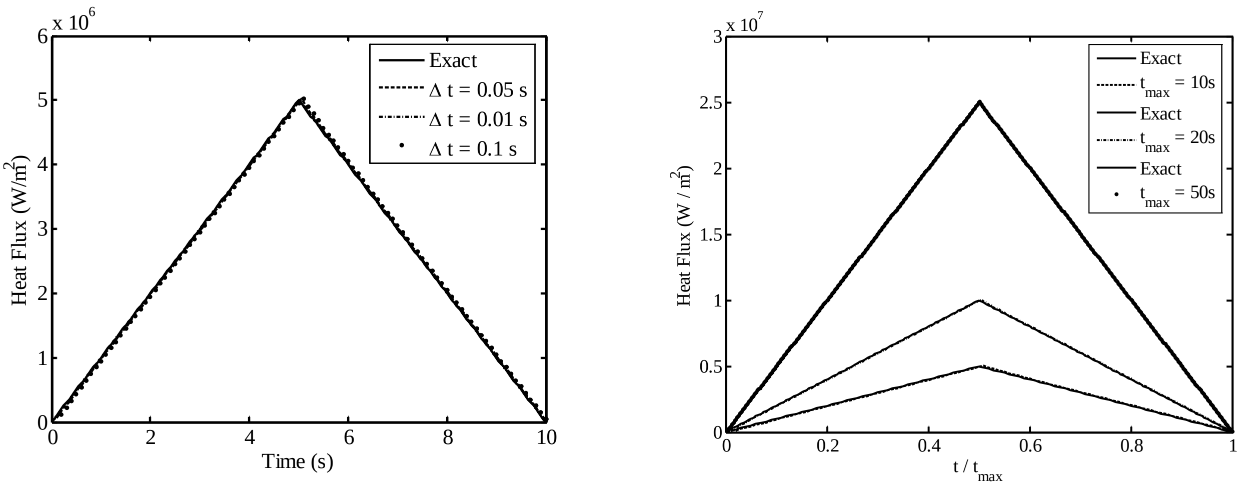
(a) (b)

Fig. 3: Comparison of estimation of step heat flux using State Space Model (SSM) and Conjugate Gradient Model (CGM) for different measurement noise levels (a) σ = 0 (b) σ = 0.02



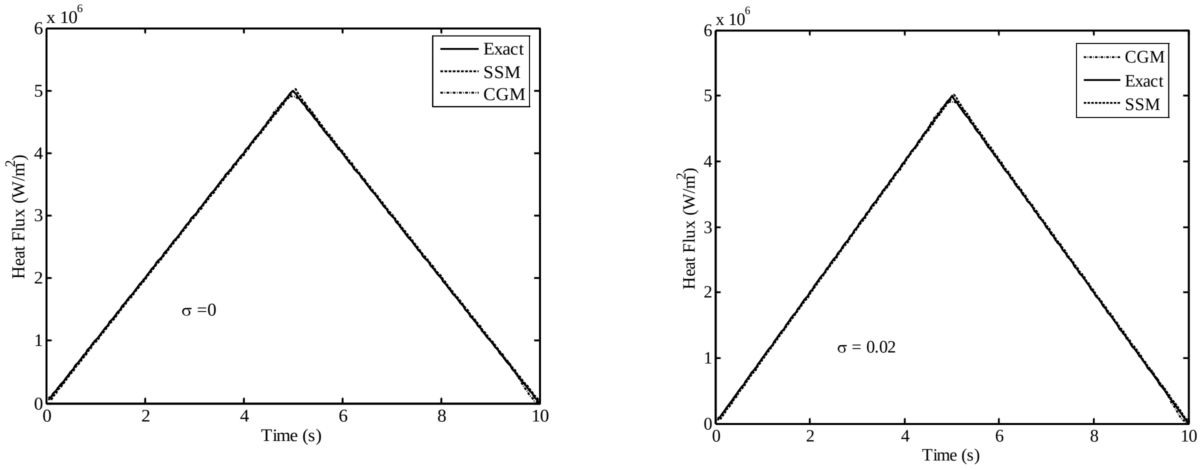
(c) (d)

Fig. 3. Comparison of estimation of step heat flux using State Space Model (SSM) and Conjugate Gradient Model (CGM) for different measurement noise levels (c) σ = 0.1 (d) σ = 1.0



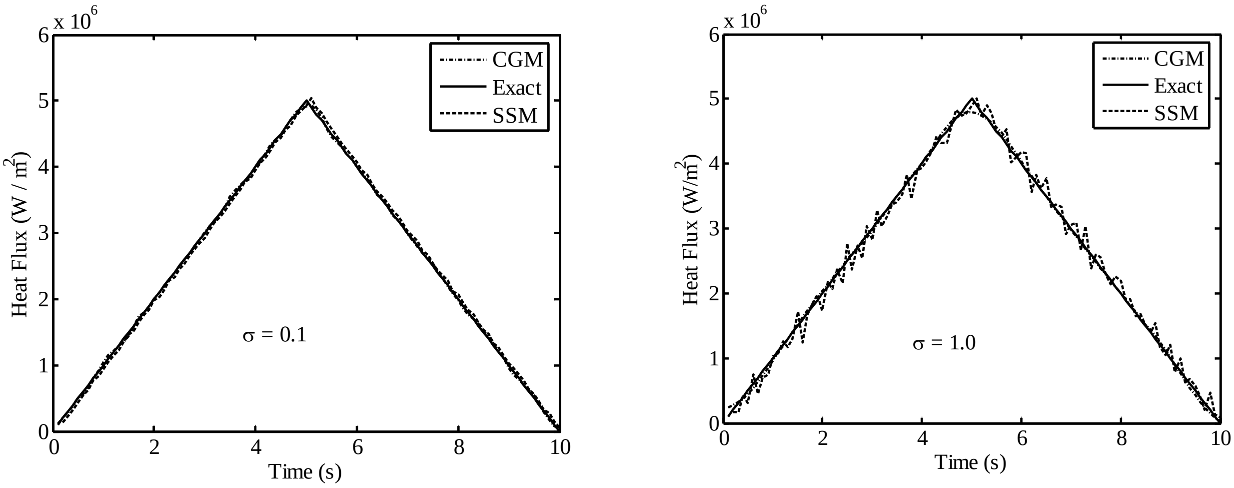
(a) (b)

Fig. 4. Effect of (a) time step size and (b) pulse duration on estimation of triangular heat flux



(a) (b)

Fig. 5: Comparison of estimation of triangular heat flux using State Space Model (SSM) and Conjugate Gradient Model (CGM) for different measurement noise levels (a) σ = 0 (b) σ = 0.02



(c) (d)

Fig. 5. Comparison of estimation of triangular heat flux using State Space Model (SSM) and Conjugate Gradient Model (CGM) for different measurement noise levels (c) σ = 0.1 (d) σ = 1.0

IV. CONCLUSION

A new method, based on state space representation of governing energy equation, is developed for estimation of boundary heat flux for heat conduction. The proposed method, being non-iterative, requires significantly less computational time than common inverse solution techniques like Conjugate Gradient Method. From the results, it is evident that, for the estimation of heat flux q (t), the present technique based on state space representation is quite accurate. The present technique is more successful in capturing sudden changes in boundary conditions than Conjugate Gradient Method, unless the noise level is extremely high.

NOMENCLATURE

A	Coefficient matrix in Eq. (2)
B	Coefficient matrix in Eq. (2)
C	Specific heat (J/kgK)
C	Coefficient matrix in Eq. (4)
D	Coefficient matrix in Eq. (4)
L	Thickness of slab (m)
N	Number of grid points
Q	Heat flux (W/m^2)
T	Time (s)
T	Temperature (K)
$\dot{T}$	dT/dt (K/s)
T	Temperature vector
U	Input vector
X	Spatial coordinate (m)
Greek Symbols	
λ	Thermal conductivity (W/mK)
ρ	Density (kg/m^3)
Subscripts	
i	i^{th} grid point
M	Measured state
N	Non-measured state

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