# Approximate analytical solution for Fractional population growth model

K.Krishnaveni [a, *], K.Kannan [b], S.Raja Balachandar [c]
[a,b,c] Department of Mathematics, School of Humanities and Sciences
SASTRA University,
Thanjavur-613401, Tamilnadu, India.
*Corresponding author
Email: [a]drkveni2012 @gmail.com, [b] kkannan@maths.sastra.edu, [c] srbala09@gmail.com

**Abstract:**

In this paper, we apply the shifted Legendre polynomial method (SLPM) to solve the fractional Volterra's model for population growth of a species in a closed system. The SLPM solution procedure for nonlinear fractional integro-differential equations is established. Moreover, the accurate analytical approximations are obtained, which are valid and convergent for different fractional orders as well as different coefficients of population growth model. This indicates the validity and great potential of the shifted Legendre polynomial method for solving nonlinear fractional integro–differential equations.

**Keywords:** Numerical approximations; shifted Legendre polynomial method; nonlinear fractional differential equation; Caputo fractional derivative.

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## 1. Introduction

The population growth [16] of a species within a closed system is characterized by a nonlinear fractional integro-differential equation in the form (Volterra's model)

$$\frac{d^{\alpha}u}{dx^{\alpha}} = au - bu^{2} - cu\int_{0}^{x} u(s)ds, \quad u(0) = \beta,\ 0 < \alpha \le 1 \qquad (1)$$

where u(x) is the scaled population of identical individuals, x denotes the time, $\alpha$ is a constant describing the order of time-fractional derivative, a>0 is the birth rate coefficient, b>0 is the crowing coefficient, and c > 0 is the toxicity coefficient [19], which denotes the essential behaviour of the population evolution before its level falls to zero in the long run, c/(ab) is a non-dimensional parameter. The last segment of (1) is a function integral represented the "total metabolism" or total amount of toxins accumulated from time zero. The individual death rate is proportional to this integral, and so the population death rate due to toxicity must include a factor $\alpha$. Since the system is closed, the presence of toxic always causes the population level to fall to zero in the long run. The more details about this model are available in [16, 18, 19 ].

Many analytical and numerical methods have been proposed to solve the classical ($\alpha = 1$) population growth model (1), namely successive approximations [16], singular perturbation [18], Adomian decomposition method [20] and other methods are available in the literature [6, 19]. In [20], ADM and Pade approximations are effectively used in the analysis to capture the essential behavior of population u(x) of identical individuals. Hang Xu[5] applied the Homotopy analysis method to solve (1) in a closed system. The detailed survey of solution methods for solving (1) is presented in [5, 17].

In the past decades, researchers have devoted considerable effort to find robust and stable numerical and analytical methods for solving FDE. There are various definitions of fractional integration and differentiation [11,14] are available, such as Grunwald Letnikov's definition and Riemann Liouville's definition. The Riemann Liouville derivative has certain disadvantages when trying to model real-world phenomena with fractional differential equations. Therefore, many researchers are used a modified fractional differential operator proposed by Caputo in his work on the theory of viscoelasticity [3].

**Definition 1.1** : The fractional derivative of f(x) in the Caputo sense is defined as

$$D_{*}^{\alpha} f(x) = J^{m-\alpha} D^{m} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_{0}^{x} (x-t)^{m-\alpha-1} f^{(m)}(t)\, dt,$$

For $m-1 < \alpha \le m,\ m \in N,\ x > 0,\ f \in C_{-1}^{m}$.

**Lemma 1.1 :** If $m-1<\alpha\le m, m\in N$ and $f\in C_\mu^m,\ m\ge -1$, then

$$D_*^\alpha J^\alpha f(x)=f(x)$$

$$J^\alpha D_*^\alpha f(x)=f(x)-\sum_{k=0}^{m-1} f^{(k)}(0^+)\frac{x^k}{k!},\ x>0$$

To obtain a numerical scheme for the approximation of Caputo derivative, we can use a representation that has been introduced by Elliott's[4];

$$D_*^q f(x)=\frac{1}{\Gamma(-q)}\int_0^x \frac{f(s)-f(0)}{(x-s)^{1+q}}ds, \qquad 0<q<1 \tag{2}$$

where the integral in equation (2) is a Hadamard finite-part integral

In general, there exists no method that yields an exact solution for FDE. Only approximate solutions can be derived using linearization or perturbation methods [1, 9, 10, 13 ].

The aim of this paper is to solve (1) numerically using the shifted Legendre polynomial method with operational matrix. Wavelet basis can be used to reduce the underlying problem to a system of algebraic equations by estimating the integrals using operational matrices [8, 12, 15, 18]. Recently the operational matrices for fractional order integration for the Haar wavelets, Chebyshev wavelets and Legendre wavelets have been developed in [9, 10, 13, 21] and the operational matrix based SLPM is discussed in[15]. SLPM reduces the given problem to a system of algebraic equations. This may lead to greater computational complexity and large storage requirements. However, the operational matrix for the shifted Legendre polynomial is structurally spare. This reduces the computational complexity of the resulting algebraic system. The application of shifted Legendre polynomial for solving differential and integral equations is thoroughly discussed in [7, 15, 21].

This paper is organized as follows: In Section 2 and 3, Properties of shifted Legendre polynomial, Function approximations, and operational matrices for fractional derivatives are presented. In Section 4, we extend the application of SLPM to construct the numerical solutions for fractional Population growth model and the approximate analytical solutions of (1) are included in the same Section. Finally, the concluding remarks are presented in section 5.

## 2. Shifted Legendre polynomials and its properties

The well-known Legendre polynomials are defined on the interval[-1,1] and can be determined with the aid of the following recurrence formulae:

$$L_{i+1}(z)=\frac{2i+1}{i+1}zL_i(z)-\frac{i}{i+1}L_{i-1}(z), \qquad i=1,2,...,$$

where $L_0(z)=1$ and $L_1(z)=z$ . In order to use these polynomials on the interval [0,1], Saadatmandi [15] utilized the so-called shifted Legendre polynomials by introducing the change of variable $z=2x-1$. Let the shifted Legendre polynomials $L_i(2x-1)$ be denoted by $P_i(x)$. Then $P_i(x)$ can be obtained as follows:

$$P_{i+1(x)}=\frac{(2i+1)(2x-1)}{(i+1)}P_i(x)-\frac{i}{i+1}P_{i-1}(x), \qquad i=1,2,..., \tag{3}$$

where $P_0(x)=1$ and $P_1(x)=2x-1.$ The analytic form of the shifted Legendre polynomial $P_i(x)$ of degree i given by

$$P_i(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2}. \tag{4}$$

Note that $P_i(0)=(-1)^i$ and $P_i(1)=1.$ The orthogonal condition is

$$\int_0^1 P_i(x)P_j(x)dx=\begin{cases}\dfrac{1}{2i+1} & for\, i=j,\\ 0 & for\ i\ne j.\end{cases} \tag{5}$$

**3. Function approximations and operational matrices**

A function y(x), square integrable in [0,1] may be expressed in terms of shifted Legendre polynomials as

$y(x)=\sum_{j=0}^{\infty} c_j P_j(x)$, where the coefficients $c_j$ are given by $c_j=(2j+1)\int_0^1 y(x)P_j(x)dx, \quad j=1,2,....$

In practice, only the first (m+1) terms of shifted Legendre polynomials are considered. Then we have

$$y(x)=\sum_{j=0}^{m} c_j P_j(x)=C^T\Phi(x), \tag{6}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by $C^T=[c_0,...,c_m],\ \Phi(x)=[P_0(x),P_1(x),...,p_m(x)]^T$. Next we present shifted Legendre Polynomial based operational matrices for fractional differentiation.

**Lemma 3.1** Let $P_i(x)$ be a shifted Legendre polynomial then

$$D^{\alpha}P_i(x)=0, \quad i=0,1,...,\lceil\alpha\rceil-1,\ \alpha>0. \tag{7}$$

In the following theorem we generalize the operational matrix of derivative of shifted Legendre polynomials given in (2) for fractional derivative.

**Theorem 3.1** Let $\Phi(x)$ be shifted Legendre vector defined in (1) and also suppose $\alpha>0$ then $D^{\alpha}\Phi(x)\cong D^{(\alpha)}\Phi(x)$ where $D^{(\alpha)}$ is the (m+1)× (m+1) operational matrix of fractional derivative of order $\alpha$ in the Caputo sense.

$$D^{(\alpha)}=\begin{pmatrix} 0 & 0 & ... & 0 \\ \vdots & \vdots & ... & \vdots \\ 0 & 0 & \cdots & 0 \\ \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,0,k} & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{\lceil\alpha\rceil}\theta_{\lceil\alpha\rceil,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,0,k} & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{i}\theta_{i,m,k} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,0,k} & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,1,k} & \cdots & \sum_{k=\lceil\alpha\rceil}^{m}\theta_{m,m,k} \end{pmatrix}$$

where $\theta_{i,j,k}$ is given by $\theta_{i,j,k}=(2j+1)\sum_{l=0}^{j}\frac{(-1)^{i+j+k+l}(i+k)!(l+j)!}{(i-k)!k!\ \Gamma(k-\alpha+1)(j-l)!(l!)^2(k+l-\alpha+1)}$

Note that in $D^{(\alpha)}$, the first $\lceil\alpha\rceil$ rows, are all zero. The proof of the lemma 3.1 and theorem 3.1 are available in [15].

**4. Solution of population growth model**

Consider the nonlinear fractional volterra integro-differential equation for population growth model (1). Set $y(x)=\int_0^x u(s)\,ds$, then $y^1(x) = u(x)$. Equation (1) implies

$$D^{\alpha+1}y=a\,Dy-b(Dy)^2-c\,yDy \ \text{ with } \ y(0)=0, y^1(0)=\beta \tag{8}$$

To solve problem (8) we approximate y(x) by the shifted legendre polynomial as

$y(x)=\sum_{j=0}^{m} c_j P_j(x)=C^T\Phi(x)$ Equation (8) becomes

$$C^T D^{\alpha+1}\Phi(x) = aC^T D\Phi(x) - b\left(C^T D\Phi(x)\right)^2 - cC^T D\Phi(x).C^T\Phi(x) . \qquad (9)$$

By tau method [2], we generate m-1 non linear equations by using

$$\int_0^1 R_m(x)P_j(x)dx = 0,\ j = 0,1,...,m-1 \qquad (10)$$

where $R_m(x) = C^T D^{\alpha+1}\Phi(x) - aC^T D\Phi(x) + b\left(C^T D\Phi(x)\right)^2 + cC^T D\Phi(x).C^T\Phi(x)$

Also, by using the initial conditions and the definition of $D^{(n)}$, we get 2 linear equations $y(0) = C^T\Phi(0) = 0$ *and* $y^1(0) = C^T D^1\Phi(0) = \beta$. These (m+1)non linear equations can be solved for unknown coefficients of the vector C by Newton 's method with aid of Matlab. Consequently y(x) can be evaluated. Next we examine the population growth model for small k and large k, k = $c/ab$

Case (i) Setting $\alpha = 1$, k=0.1(ie, a=b=c=10), $\beta = 0.1$; By applying SLPM with M=8, the approximate solution as $y(x) = \sum_{j=0}^{8} c_j P_j(x) = C^T\Phi(x)$ with 9 unknown coefficients of vector C. We obtain values of polynomial coefficients c0 =53.679303, c1 =209.235613, $c_2$=234.999235, c3=176.589083, c4=93.405479, c5=34.415688, c6=8.436652, c7=1.236478, c8=0.082073 and the approximate solution of (1) is u(x) = 0.1 + 0.9x +3.55$x^2$ + 6.31666667$x^3$ – 5.5375$x^4$ – 63.70916667$x^5$ – 156.0804167$x^6$ – 18.47323411$x^7$ + 1056.288569$x^8$ which is the same as in [17].

Case (ii) Setting $\alpha = 0.5$, k=0.1(ie, a=b=c=10), $\beta = 0.1$ we obtain the approximate solution of (1) is u(x) = 0.1 + 1.01554x +7.2$x^2$ + 35.4964$x^3$ + 90.422$x^4$ – 321.158$x^5$ – 5346.32$x^6$ – 32307.8$x^7$ – 82694.8$x^8$ , is the same as in [17].

Case (iii) For k=1(a=b=c=1) with $\alpha$ =1 , u(x) = 0.1 + 0.09x + 0.031$x^2$+0.0010666$x^3$ – 0.0032275$x^4$ – 0.001238666$x^5$ + 0.00068208$x^6$ +0.000057765$x^7$-0.0000067618$x^8$.

Also the numerical solutions of u(x) for M=8 with $\alpha$ = 0.25,0.5,0.75, 1 and k=1are plotted in Fig 1.

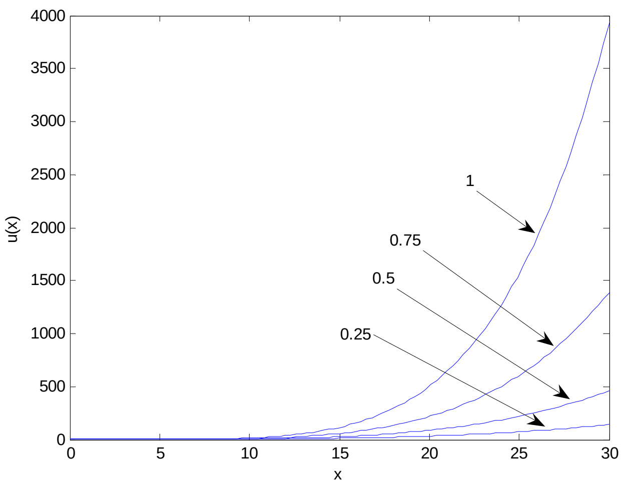


Fig 1. Comparison of u(x) for M=8 with $\alpha$ = 0.25, 0.5, 0.75, 1 when k=1.

## 5. Conclusion

In this work, operational matrix based shifted Legendre polynomial has been used to find the numerical solution of fractional population growth model. The properties of shifted Legendre polynomials and Caputo derivative are used to reduce the problem to the solution of linear/ nonlinear algebraic equations with appropriate coefficients which provide exact solutions for all the chosen problems, Moreover, the computational complexity of solving the system of algebraic equation is very less because the matrix is sparse. This method is expected to be further employed to solve other similar problems in fractional calculus.

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