The Rate Hahn Sequence Space

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***Abstract*— This paper is devoted to the rate space of Hahn sequence space. It is denoted by h_π. Some properties of h_π are investigated. Recent work on rate spaces is given in [2] and [3].The space h_π with sequences in a Banach algebra is propounded.**

Keywords- Sequence spaces, β-dual, BK-space, AK-space, Hahn sequence space.

I. INTRODUCTION

A Notation and Definition (See [5])

We consider sequences in a Banach algebra A with norm$\|\quad\|$.

Let $a = (a_k)$ with $a_k \in A$.

$|||a|||$ is the norm of the sequence.

$$l = \{a: \textstyle\sum_{k=1}^{\infty} \|a_k\| < \infty\}$$

$$bv = \{a: \textstyle\sum_{k=1}^{\infty} \|\Delta a_k\| < \infty\}$$

$$c_0 = \left\{a: \lim_{k \to \infty} a_k = 0\right\}$$

$$l_\pi = \left\{a: \textstyle\sum_{k=1}^{\infty} \left\|\frac{a_k}{\pi_k}\right\| < \infty\right\}$$

$$bv_0 = \{a: \textstyle\sum_{k=1}^{\infty} \|\Delta a_k\| < \infty, \Delta a_k = a_k - a_{k+1}, \lim_{k\to\infty} a_k = 0\}$$

h_π is the vector space of all sequences $\{x_k\}$ such that $\sum_{k=1}^{\infty} k \left\|\frac{x_k}{\pi_k} - \frac{x_{k+1}}{\pi_{k+1}}\right\| < \infty$.

It is a Banach space as shown in theorem (2.1).

$$cs = \{a: \textstyle\sum_{k=1}^{\infty} a_k \ \ exists\}$$
$$\sigma_0 = \{a: n^{-1} \textstyle\sum_{k=1}^{\infty} a_k \to 0 (n \to \infty)\}$$
$$\sigma_\infty = \left\{a: \sup_{(n)} n^{-1} \|\textstyle\sum_{k=1}^{n} a_k\| < \infty\right\}$$

The space h is found in [4].

B Result [1]

Let (X,p) and(Y,q) be semi-normed spaces and T: (X,p)→(Y,q) be an isometric isomorphism (that is T is an isomorphism and satisfies q(T(x))=p(x) for each $x \in X$). Then (X,p) is complete if and only if (Y,q) is complete. In particular, (X,p) is a Banach space if and only if (Y,q) is a Banach space.

The following results are established.

II. MAIN RESULTS

Theorem (2.1):

h_π is a Banach space with the norm $|||x||| = \left\{\|x_0\| + \sum_{(k)} k \left\|\frac{x_k}{\pi_k} - \frac{x_{k+1}}{\pi_{k+1}}\right\| < \infty\right\}$.

Proof:

Obviously $\sum^{-1}: \left(h_\pi, |||\quad|||_{h_\pi}\right) \to (l, |||\quad|||_1), (x_k) \to (x_k - x_{k-1})$ where $x_{-1} \coloneqq 0$ is an isometric isomorphism. Thus $\left(h_\pi, |||\quad|||_{h_\pi}\right)$ is a Banach space because of following result.

Theorem (2.2):[1]

(l_π, d_1) is a complete metric space with the metric

$$d_1(x,y) = |||x-y|||\forall x = \left(\frac{x_k}{\pi_k}\right) \; and \; y = \left(\frac{y_k}{\pi_k}\right) \; in \; l_\pi.$$

Proof:

To prove that (l_π, d_1) is complete, we assume that

$(x^{(n)})$ with $x^{(n)} = (x_k^{(n)})$ is a Cauchy sequence in (l_π, d_1).

Since $l_\pi \subset l_{\frac{1}{\pi}}^{\infty}$ and because for all $x = \left(\frac{x_k}{\pi_k}\right), y = \left(\frac{y_k}{\pi_k}\right) \in l_\pi$ the inequality $d_\infty(x,y) = \sup_{k \in \mathbb{N}^0} \|x_k - y_k\| \leq \sum_{k=0}^{\infty} \|x_k - y_k\| = d_1(x,y)$ holds, $(x^{(n)})$ is also a Cauchy sequence in $\left(l_{\frac{1}{\pi}}^{\infty}, d_\infty\right)$.

However , $\left(l_{\frac{1}{\pi}}^{\infty}, d_\infty\right)$ is complete that is $(x^{(n)})$ converges to some $x = (x_k) \in l_{\frac{1}{\pi}}^{\infty}$ relative to d_∞.

We now show that $x \in l_\pi$ and $\lim_{n \to \infty} d_1\left(x^{(n)}, x\right) = 0$ which proves that $(x^{(n)})$ converges $(to\ x)$ in (l_π, d_1).
For any given $\varepsilon > 0$ we choose an $n_0 \in \mathbb{N}$ such that

$$d_1\left(x^{(n)}, x^{(\upsilon)}\right) < \frac{\varepsilon}{2} \quad (n.\upsilon \geq n_0) \qquad (1)$$

Since $\lim_{n \to \infty} d_\infty\left(x^{(\upsilon)}, x\right) = 0$ for each $N \in \mathbb{N}$

we can choose a $\upsilon_N \in \mathbb{N}$ with $\upsilon_N \geq n_0$ such that

$$d_\infty\left(x^{(\upsilon)}, x\right) < \frac{1}{N}\frac{\varepsilon}{2}(\upsilon \geq \upsilon_N) \; . \qquad (2)$$

Thus we get for each given $N \in \mathbb{N}$ and all $n \geq n_0$ and every $\upsilon \geq \upsilon_N$ the inequalities

$$\sum_{k=0}^{N} \left\|x_k^{(n)} - x_k\right\| \leq \sum_{k=0}^{N} \left\|x_k^{(n)} - x_k^{(\upsilon)}\right\| + \sum_{k=0}^{N} \left\|x_k^{(\upsilon)} - x_k\right\|$$

$$\leq d_1\left(x^{(n)}, x^{(\upsilon)}\right) + N d_\infty\left(x^{(\upsilon)}, x\right)$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \; (on\ account\ of\ (1) and\ (2)).$$

Hence, since $N \in \mathbb{N}$ is given, we have $\forall \varepsilon > 0\ \exists\ n_0 \in \mathbb{N}\ \forall n \geq n_0 : \sum_{k=0}^{\infty} \left\|x_k^{(n)} - x_k\right\| \leq \varepsilon$ which implies $x^{(n)} - x \in l_\pi$ for every $n \geq n_0$, thus $x \in l_\pi$ and $\lim_{n \to \infty} d_1\left(x^{(n)}, x\right) = 0$ that is $(x^{(n)})$ converges to x.

Theorem-(2.3):

$h_\pi = l_\pi \cap \int(bv)_\pi = l_\pi \cap \int(bv_0)_\pi.$

Proof:

For $k = 1,2, \ldots$

$$k\frac{\Delta a_k}{\pi_k} = \frac{a_{k+1}}{\pi_{k+1}} + \Delta\left(\frac{k a_k}{\pi_k}\right) \qquad (3)$$

Hence $a \in h$ implies

$$\infty > \sum_{k=1}^{\infty} k \left\|\frac{\Delta a_k}{\pi_k}\right\| \geq \sum_{k=1}^{\infty} \left\|\Delta\left(\frac{k a_k}{\pi_k}\right)\right\| - \sum_{k=1}^{\infty} \left\|\frac{a_{k+1}}{\pi_{k+1}}\right\|.$$

The last series is convergent since $h_\pi \subset l_\pi$.

Hence also $\sum_{k=1}^{\infty} \left\|\Delta\left(\frac{k a_k}{\pi_k}\right)\right\| < \infty$ and therefore $h_\pi \subset l_\pi \cap \int(bv)_\pi$. (4)

Conversely (3) implies for $a \in l_\pi \cap \int(bv)_\pi$, $\infty > \sum_{k=1}^{\infty} \left\|\frac{a_{k+1}}{\pi_{k+1}}\right\| + \sum_{k=1}^{\infty} \left\|\Delta\left(\frac{k a_k}{\pi_k}\right)\right\| \geq \sum_{k=1}^{\infty} k \left\|\frac{\Delta a_k}{\pi_k}\right\|$ and $\lim_{k \to \infty} \frac{a_k}{\pi_k} = 0$. Thus $l_\pi \cap \int(bv)_\pi \subset h_\pi$. (5)

From (4) and (5) we get $h_\pi = l_\pi \cap \int(bv)_\pi$.

Theorem(2.4):

$h_\pi = (\sigma_\infty)_{\frac{1}{\pi}}^{\beta}.$

Proof:

We know that $h_\pi^\beta = (\sigma_\infty)_{\frac{1}{\pi}}$. Now it is enough to show that h_π is a β-dual K$\ddot{o}$the space. In fact we know that $h_\pi = l_\pi \cap \int(bv)_\pi$ and as is well known that $l_\pi = (c_0)_{\frac{1}{\pi}}^\beta$ and $\int(bv)_\pi = \big(d(cs)\big)_{\frac{1}{\pi}}^\beta$ since $(bv)_\pi = (cs)_{\frac{1}{\pi}}^\beta$.

Theorem-(2.5):

h_π is a BK-space with AK.

Proof:

l_π and $\int(bv_0)_\pi$ with the norms $|||a||| = \sum_{k=1}^{\infty} \left\|\frac{a_k}{\pi_k}\right\|$ and $|||a||| = \sum_{k=1}^{\infty} \left\|\Delta\left(\frac{ka_k}{\pi_k}\right)\right\|$ respectively are BK-spaces with AK. Hence by the known result of the intersection of two BK-spaces with AK is again a BK-space with AK, the result follows.

Theorem-(2.6):

(i) $h'_\pi = (\sigma_\infty)_{\frac{1}{\pi}}$ (ii) $(\sigma_0)'_\pi = (h)_{\frac{1}{\pi}}$ where h' is the conjugate space of h.

Proof:

(i) We know that h_π is a BK-space with AK. Also we know that $h_\pi^\beta = (\sigma_\infty)_{\frac{1}{\pi}}$.
But $h_\pi^\beta = h'_\pi$.
Therefore we get $h'_\pi = (\sigma_\infty)_{\frac{1}{\pi}}$.

(ii) It is known that σ_0 with the norm $|||a||| = \sup_{(n)} n^{-1} \left\|\sum_{k=1}^{n} \frac{a_k}{\pi_k}\right\|$ is a BK-space with AK.
Hence again by known result $(\sigma_0)_{\frac{1}{\pi}}^\beta = (\sigma_0)'_{\frac{1}{\pi}}$. It remains to be shown that $(\sigma_0)_{\frac{1}{\pi}}^\beta = h_\pi$.
But $\int(\sigma_0)_\pi = \int((c_0)_\pi + (cs)_\pi)$.
Hence $(\sigma_0)_\pi = (c_0)_\pi + \big(d(cs)\big)_\pi$ and this implies $(\sigma_0)_{\frac{1}{\pi}}^\beta = (c_0)_{\frac{1}{\pi}}^\beta \cap \big(d(cs)\big)_{\frac{1}{\pi}}^\beta = h_\pi$ since $(c_0)_{\frac{1}{\pi}}^\beta = l_\pi$ and $\big(d(cs)\big)_{\frac{1}{\pi}}^\beta = \int(bv)_\pi$.

Theorem-(2.7):

Let h_π be a subspace of a normed space l_π. If h_π is complete , then h_π is closed.

Proof:

Let x be a limit point of h_π.
Then every open sphere centred on x contains points (other than x) of h_π.
In particular, the open sphere $S\left(x, \frac{1}{n}\right)$ where n is a positive integer contains a point x_n of h_π other than x. Thus $\{x_n\}$ is a sequence in h_π such that $|||x_n - x||| < \frac{1}{n}$ $\forall\, n$.

$\Rightarrow$ $\lim_{n\to\infty} x_n = x$ in l_π.

$\Rightarrow$ $\{x_n\}$ is a Cauchy sequence in l_π and hence in h_π.

But h_π being complete, it follows that $x \in h_\pi$. This proves that h_π is closed.

Theorem-(2.8):

Let h_π be a subspace of a Banach space l_π. If h_π is closed , then h_π is complete.

Proof:

Let $\{x_n\}$ be a Cauchy sequence in h_π. Then it is so in l_π. But l_π being complete, $\exists\, x \in l_\pi$ such that $x_n \to x$. Either $x \in h_\pi$, then we are done, or each neighbourhood of x contains points x_n $(\neq x)$. As such, x is a limit point of h_π. But h_π being closed, it follows that $x \in h_\pi$. Hence the result is proved.
Thus we obtain h_π as a subspace of a Banach space l_π. Then h_π is complete if and only if h_π is closed.

III. EXAMPLES

1. Consider the space φ_π of sequences $x = \left(\frac{\xi_1}{\pi_1}, \frac{\xi_2}{\pi_2}, \ldots \frac{\xi_n}{\pi_n}, 0, \ldots\right)$ in $\mathbb{K}$ where $\frac{\xi_n}{\pi_n} \neq 0$ for only finitely many values of n. Clearly $\varphi_\pi \subset (c_0)_\pi \subset l_\pi^\infty$ and $\varphi_\pi \neq (c_0)_\pi$.

   But $(c_0)_\pi$ is the closure of φ_π in $(l_\pi^\infty, |||.|||_\infty)$.
   Thus φ_π is not closed in l_π^∞ and hence φ_π is an incomplete normed space equipped with the

norm induced by the norm $|||.|||_\infty$ $on\ l_\pi^\infty$.

2. For every real number $p \geq 1$we have $\varphi_\pi \subset l_\pi^p \subset (c_0)_\pi$.
   But $(c_0)_\pi$ is the closure of
   $l_\pi^p\ in\ (c_0)_\pi\ and l_\pi^p \neq (c_0)_\pi.\ Thus\ l_\pi^p\ is\ not\ closed\ in\ (c_0)_\pi\ and\ hence\ l_\pi^p$ is an incomplete normed space when endowed with the norm induced by $|||.|||_\infty$ $on\ (c_0)_\pi$.
3. For every real number $p = 1$ we have $\varphi_\pi \subset l_\pi^p$.
   But l_π^p is the closure of φ_π in $\left(l_\pi^p, |||.|||_p\right)$ and $\varphi_\pi \neq l_\pi^p$.
   Thus φ_π is not closed in l_π^p and hence φ_π is an incomplete normed space endowed with the norm induced by $|||.|||_p$ on l_π^p.

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