# Intuitionistic Fuzzy Lattices and Intuitionistic Fuzzy Boolean Algebras

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***Abstract*—The concept of lattice plays a significant role in mathematics and other domains where order properties play an important role. In fact, a Boolean algebra, which is a special case of a Lattice, is of fundamental importance in Computer science. The notion of Fuzzy sets as an extension of Crisp sets is a model to handle uncertainty in data. Following it, the notions of fuzzy lattices have been introduced by many researchers [1, 11]. One of the approaches is the introduction of a fuzzy partially ordered relation on a crisp set which satisfies the lattice property. Intuitionistic fuzzy sets [2] are generalisations of the notion of fuzzy sets. So far the notion of Intuitionistic fuzzy lattices has neither been introduced nor studied. In this paper, we introduce the notions of intuitionistic fuzzy lattices, intuitionistic fuzzy Boolean algebra and study their properties. In this process, we provide an alternate definition of antisymmetric property of intuitionistic fuzzy relations and compare it with the existing ones.**

**Keyword-** Intuitionistic fuzzy set, Intuitionistic fuzzy relation, Intuitionistic fuzzy lattice, Intuitionistic fuzzy Boolean algebra

## I. INTRODUCTION

The notion of fuzzy sets [ 13] and that of fuzzy relations [14 ] have been introduced as extensions of crisp sets and crisp relations to model uncertainty in data and information. The concepts of intuitionistic fuzzy sets [2] and intuitionistic fuzzy relations [3, 5, 6, 7, 8, 9] are further extensions in this direction and these notions generalise the notions of fuzzy sets and fuzzy relations respectively. The special types of relations like equivalence relations and partially ordered relations have important applications in mathematics. The notions of lattices in general and that of Boolean algebra in particular have important role in Computer science and rely on partially ordered relations. The concept of fuzzy lattice has been introduced in many ways [1, 11]. Perhaps the most natural approach among all these by defining a partially ordered fuzzy relation over a crisp set is due to Tripathy et al [12]. Most importantly, besides the study of several special types of fuzzy lattices, they have introduced the concept of fuzzy Boolean algebra. The study of various types of intuitionistic fuzzy relations has been done in literature [3, 5, 6, 7, 8, 9]. In this paper we introduce the notions of intuitionistic fuzzy lattice, intuitionistic fuzzy Boolean algebra and study their properties and also properties of some special type of such lattices.

## II. FUZZY LATTICES AND FUZZY BOOLEAN ALGEBRAS

### A. *Fuzzy Lattices*

We first introduce some preliminary definition, which shall be used to define fuzzy lattices.

**Definition 2.1.1:** A fuzzy binary relation R on a set X is a fuzzy partial ordering if and only if it is fuzzy reflexive, fuzzy antisymmetric and fuzzy transitive under any fuzzy transitivity.

Here, we shall confine ourselves to the max-min transitivity only. There are several other definitions of fuzzy transitivity.

**Definition 2.1.2:** Let R be a fuzzy partial ordering defined on a set X. Then, for any element $x \in X$, we associate two fuzzy sets. The first one is the dominating class of x( [10]) denoted by $R_{\geq[x]}$ and is defined as

$$(2.1.1) \qquad R_{\geq[x]}(y) = R(x, y),$$

where $y \in X$. In other words, the dominating class of x contains the members of X to the degree to which they dominate x.

The second one is, the class dominated by x ([10]), denoted by $R_{\leq[x]}$ and is defined as

$$(2.1.2) \qquad R_{\leq[x]}(y) = R(y, x),$$

where y ∈ X. We have, the class dominated by x contains the elements of X to the degree to which they are dominated by x.

**Definition 2.1.3:** An element x ∈ X is undominated if and only if R(x, y) = 0 for all y ∈ X and x ≠ y. X is said to be undominating if and only if R(y, x) = 0 for all y ∈ X and x≠y.

**Definition 2.1.4:** For a crisp A of X on which a fuzzy partial ordering R is defined, the fuzzy upper bound set of A is the fuzzy set denoted $U\left(R,\ A\right)$ and is defined by ([10])

$$(2.1.3) \qquad U(R, A) = \bigcap_{x \in A} R_{\geq[x]}\ .$$

**Definition 2.1.5:** For a crisp subset A of a set X on which a fuzzy partial ordering R is defined, the fuzzy lower bound set for A is the fuzzy set denoted by L(R,A) and is defined by

$$(2.1.4) \qquad L(R, A) = \bigcap_{x \in A} R_{\leq[x]}\ .$$

**Definition 2.1.6:** The least upper bound of A with respect to the fuzzy partial ordering relation R is a unique element x in $U(R, A)$ such that

$$(2.1.5) \qquad U(R, A)(x) > 0 \ and \ R(x, y) > 0\ ,$$

for all elements y in the support of $U(R, A)$.

**Note 2.1.1:** If there are two such elements x and y then by (2.1.5) we have $R(x, y) > 0 \ and \ R(y, x) > 0.$ So, by the antisymmetric property x = y.

**Definition 2.1.7:** The greatest lower bound with respect to the fuzzy partial ordering relation R is a unique element x in L(R, A) such that

$$(2.1.6) \qquad L(R, A)(x) > 0 \ and \ R(y, x) > 0\ ,$$

for all elements in the support of L(R,A).

**Definition 2.1.8:** A crisp set X on which a fuzzy partial ordering R is defined is said to be a fuzzy lattice if and only if for any two element set {x, y} in X, the least upper bound (lub) and the greatest lower bound (glb) exist in X.

We denote the lub of {x, y} by $x \vee y$ and the glb of {x, y} by $x \wedge y$.

Many properties of fuzzy lattices defined this manner are established in [12].

*B. Fuzzy Boolean algebra*

Boolean algebras have an important role in the application areas like computer since. It is a special kind of lattice. So, obviously one can expect fuzzy Boolean algebra to be considered as a special case of fuzzy lattice. We define it below ([12]).

**Definition 2.2.1:** A complemented distributive fuzzy lattice is called a fuzzy Boolean algebra.

**Definition 2.2.2:** Let $\underline{B} = \left(B, \wedge, \vee\ , 0, 1, '\right)$ be a fuzzy Boolean algebra. For any two elements a and b in B, we define the operation ring sum denoted by $\oplus$ as

$$(2.2.1) \qquad a \oplus b = \left(a \wedge b'\right) \vee \left(a' \wedge b\right).$$

**Definition 2.2.3:** In any Fuzzy Boolean algebra $\underline{B} = \left(B, \wedge\ , \vee, 0, 1, '\right)$, a ring product $\odot$ is defined by

$$(2.2.2) \qquad a \odot b = a \wedge b, \ \ \forall\ a, b \in B$$

**Definition 2.2.4:** A complemented distributive fuzzy lattice B with the binary operation $\oplus \ and \odot$ is a fuzzy Boolean ring with identity 1.

**Definition 2.2.5:** Let $\left(L, \wedge\ , \vee\right)$ be a fuzzy lattice with a lower bound 0. An immediate successor of 0 is called an atom.

**Definition 2.2.6:** Let $(L, \wedge , \vee)$ be a fuzzy lattice with an upper bound 1. An immediate predecessor of 1 is called an antiatom.

**Definition 2.2.7:** Let $(L, \wedge , \vee)$ be a fuzzy lattice. An element a $\in$ L is said to be join irreducible if

(2.2.3) $$a = a_1 \vee a_2 \Rightarrow a = a_1 \text{ or } a = a_2.$$

**Definition 2.2.8:** An element a in a fuzzy lattice $(L, \wedge , \vee)$ is said to be meet irreducible if

(2.2.4) $$a = a_1 \vee a_2 \Rightarrow a = a_1 \text{ or } a = a_2.$$

III. INTUITIONISTIC FUZZY LATTICES

We generalise the notion of fuzzy lattices introduced by Tripathy and Chaudhury ([12]) to the setting of intuitionistic fuzzy lattices in this section. As is well known fuzzy sets are special cases of intuitionistic fuzzy sets. Hence intuitionistic fuzzy sets have better modeling power than the fuzzy sets. Similarly, intuitionistic fuzzy lattices are more general than the fuzzy lattices. First we introduce some definitions which we define below:

**Definition 3.1:** An intuitionistic fuzzy relation is an intuitionistic fuzzy subset of X $\times$ Y; that is R given by

(3.1) $$R = \{< (x, y), \mu_R(x, y), \nu_R(x, y) > / x \in X, y \in Y\},$$

where

$$\mu_R, \nu_R : X \times Y \rightarrow [0,1],$$

satisfy the condition $0 \le \mu_R(x, y) + \nu_R(x, y) \le 1,$ for every $(x, y) \in X \times Y.$

**Definition 3.2:** We say that an intuitionistic fuzzy relations R is

(3.2) reflexive, if for every x $\in$ X, $\mu_R(x, x) = 1 \text{ and } \nu_R(x, x) = 0.$

(3.3) irreflexive, if for some $x \in X, \mu_R(x, x) \ne 1 \text{ or } \nu_R(x, x) \ne 0.$

(3.4) antireflexive, if for every $x \in X, \mu_R(x, x) = 0 \text{ and } \nu_R(x, x) = 1.$

(3.5) symmetric, if for every $(x, y) \in X \times X$

$$\mu_R(x, y) = \mu_R(y, x) \text{ and } \nu_R(x, y) = \nu_R(y, x).$$

In the opposite case we say R is asymmetric.

(3.6) Perfect antisymmetric if for every $(x, y) \in X \times X$ with $x \ne y$ and

$\mu_R(x, y) > 0 \text{ or } (\mu_R(x, y) = 0 \text{ and } \nu_R(x, y) < 1)$ then

$\mu_R(x, y) = 0 \text{ and } \nu_R(x, y) = 1.$

(3.7) transitive if $R \circ R \subseteq R$ , where o is max-min and min-max composition, that is

$\mu_R(x, z) \ge \max_y \left[ \min\{\mu_R(x, y) , \mu_R(x, y)\} \right]$ and

$\nu_R(x, z) \le \min_y \left[ \max \{\nu_R(x, y) , \nu_R(y, z)\} \right].$

**Note 3.1:** The definition of perfect antisymmetric given in (3.6) is equivalent to the following: an intuitionistic fuzzy relation R is perfect antisymmetric if for every $(x, y) \in X \times X$

(3.8) $$(\mu_R(x, y) > 0 \text{ and } \mu_R(y, x) > 0) \text{ or } (\nu_R(x, y) < 1 \text{ and } \nu_R(y, x) < 1) \Rightarrow x = y.$$

**Observation 3.1:** First we see that when X is a fuzzy set, $\nu_R(x, y) = 1 - \mu_R(x, y)$ and $\nu_R(y, x) = 1 - \mu_R(y, x).$ So that the second half of condition (3.8) reduces to the first part and so we get the corresponding definition of antisymmetry of fuzzy set.

**Observation 3.2:** Next, we note that when first half of condition (3.8) is true the second half follows. But, when the second half holds, we get that $\mu_R(x, y) \ge 0 \text{ and } \mu_R(y, x) \ge 0.$ So, the first half does not follow. To be precise, the second half covers the cases $\mu_R(x,y)=0 \; or \; \mu_R(y,x)=0.$ So, we cannot do away with any of the parts in (3.8).

**Observation 3.3:** Finally, we shall show that (3.6) and (3.8) are equivalent.

(i) Proof of $(3.8)\Rightarrow(3.6)$

We have (3.8) is equivalent to

(3.9) $$(x\neq y)\Rightarrow\neg(\mu_R(x,y)>0 \; and \; \mu_R(y,x)>0) \\ and \; \neg(\nu_R(x,y)<1 \; and \; \nu_R(y,x)<1).$$

In (3.9) $'\neg'$ is the Boolean negation. In addition if $\mu_R(x,y)>0 \; then$ from the first expression of RHS we get $\mu_R(y,x)=0.$

From the second expression on the RHS we get either both $\nu_R(x,y)=1 \; and \; \nu_R(y,x)=1$ or one of these is equal to 1 and the other is not. But in this case as $\mu_R(x,y)>0$, we cannot have $\nu_R(x,y)=1.$ So, we have $\nu_R(y,x)=1.$ Hence, in any case we get $\nu_R(x,y)=1.$

On the other hand suppose $\mu_R(x,y)=0 \; and \; \nu_R(x,y)<1.$ Then from the second expression on the RHS of (3.9), we must have $\nu_R(y,x)=1.$ Hence, $\mu_R(y,x)=0.$

(ii)Proof of $(3.6)\Rightarrow(3.8)$

Suppose $x\neq y.$ Then we have two cases.

Case-1: $\mu_R(x,y)>0.$

In this case we have $\mu_R(y,x)=0 \; by \; (3.6).$ Also $\nu_R(y,x)=1.$ So that RHS of (3.9) is true.

Case-2: $\mu_R(x,y)=0.$ Then either $\nu_R(x,y)=1 \; or \; \nu_R(x,y)<1.$

If $\nu_R(x,y)=1$ then $\mu_R(x,y)=0.$ So, RHS of (3.9) is true.

On the other hand if $\nu_R(x,y)<1$ then by (3.6) $\mu_R(y,x)=0$ and $\nu_R(y,x)=1.$ So, RHS of (3.9) is true.

**Definition 3.3:** An intuitionistic fuzzy relation R on X is said to be all intuitionistic fuzzy partially ordered relation if R is reflexive, perfect antisymmetric and transitive; that is (3.2), (3.6) and (3.7) hold.

**Definition 3.4:** Let X be a set with an intuitionistic fuzzy partial ordered relation ‘R’ defined over it. Then (X, R) is called an intuitionistic fuzzy partially ordered set.

**Definition 3.5:** When a intuitionistic fuzzy partial ordering in defined on a set X, two intuitionistic fuzzy sets are associated with each element x in X. The first is called the dominating class of x. We denote it by $R_{\geq[x]}$ and is defined by

(3.10) $$y\in\mu_{R\geq[x]} \; iff \; \mu_R(x,y)>0 \; or \; \{\mu_R(x,y)=0 \; and \; \nu_R(x,y)<1\}.$$

The second is called the class dominated by x. we denote it by $R_{\leq[x]}$ and is defined by

(3.11) $$y\in\mu_{R\leq[x]} \; iff \; \mu_R(y,x)>0 \; or \; \{\mu_R(y,x)=0 \; and \; \nu_R(y,x)<1\}.$$

**Definition 3.6:** An element $x\in X$ is undominated if and only if

(3.12) $$\mu_R(x,y)=0 \;\; and \;\; \nu_R(x,y)=1 \text{ for all } y\in X \text{ and } y\neq x$$

An element $x\in X$ is undominating if and only if

(3.13) $$\mu_R(y,x)=0 \; and \; \nu_R(y,x)=1 \; for \; all \; y\in X \; and \; y\neq x.$$

**Definition 3.7:** For a crisp subset A of a set X on which an intuitionistic fuzzy partial ordering R is defined, the intuitionistic fuzzy upper bound for A in the intuitionistic fuzzy set denoted by $U(R,A)$ and is defined by

(3.14) $$U(R, A) = \bigcap_{x \in A} R_{\geq [x]}$$

when '$\bigcap$' denotes intersection of intuitionistic fuzzy sets.

**Definition 3.8:** Let A be an intuitionistic fuzzy set. Then by the support set of A, We mean all elements 'x' for which $\mu_A(x) > 0 \;\; or \; \{\mu_A(x) = 0 \; and \; \nu_A(x) < 1\}$.

**Definition 3.9:** For a crisp subset A of a set X on which an intuitionistic fuzzy partial ordering R is defined, the intuitionistic fuzzy lower bound for A is the intuitionistic fuzzy set denoted by L(R,A) and is defined by

(3.15) $$L(R, A) = \bigcap_{x \in A} R_{\leq [x]} .$$

**Definition 3.10:** The least upper bound of A with respect to the intuitionistic fuzzy partial ordering relation R is a unique element x in support set of $U(R, A)$ such that

(3.16) For all other y in support set of $U(R, A)$ $\mu_R(x, y) > 0 \; or \{\mu_R(x, y) = 0 \; and \; \nu_R(x, y) < 1\}$.

**Definition 3.11:** The greatest lower bound of A with respect to the intuitionistic fuzzy partial ordering relation R is a unique element x in the support set of L(R, A) such that

(3.17) For all other y in the support set of L(R, A) $\mu_R(y, x) > 0 \; or \{\mu_R(y, x) = 0 \; and \; \nu_R(y, x) < 1\}$.

**Note 3.2:** The uniqueness of x in definitions 3.10 and 3.11 follows from the intuitionistic fuzzy antisymmetric property of R. We provide the proof for 3.10. The case of 3.11 is similar.

Suppose that there are two such elements x and z. Then there are three cases.

**Case (i)** If both x and z satisfy $\mu_{L(R,A)}(x) > 0 \; and \; \mu_{L(R,A)}(z) > 0$ then $\mu_R(x, z) > 0 \; and \; \mu_R(z, x) > 0.$ So, by antisymmetric property x = z.

**Case (ii)** If both satisfy $\mu_{L(R,A)}(x) = 0, \nu_{L(R,A)}(x) < 1 \; and \; \mu_{L(R,A)}(z) = 0, \nu_{L(R,A)}(z) < 1$ then from $\nu_{L(R,A)}(x) < 1 \; and \; \nu_{L(R,A)}(z) < 1$ we get x = z.

**Case (iii)** If $\mu_{L(R,A)}(x) > 0$, $\mu_R(z, x) > 0$ $(which \; implies \; \nu_R(z, x) < 1)$ and $\mu_{L(R,A)}(z) = 0, \nu_{L(R,A)}(z) < 1$ (which implies $\nu_R(x, z) < 1$). So, from $\nu_R(z, x) < 1 \; and \; \nu_R(z, x) < 1$ we get by antisymmetry that x = z.

**Definition 3.12:** A crisp set X on which a intuitionistic fuzzy partial ordering R is defined is said to be an intuitionistic fuzzy lattice if and only if for any two element set {x, y} in X, the least upper bound (lub) and greatest lower bound (glb) exist in X.

We denote the lub of {x, y} by $x \vee y$ and glb of {x, y} by $x \wedge y$.

**Example 3.1:** Let *X* = {a, b, c, d, e}. We define an intuitionistic fuzzy relation R over X, given by the matrix:

$$\begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{bmatrix} (1.0) & (\cdot 7, \cdot 2) & (0,1) & (0,1) & (0,1) \\ (0, \cdot 8) & (1,0) & (0,1) & (\cdot 9, \cdot 1) & (0, \cdot 8) \\ (\cdot 5, \cdot 3) & (\cdot 7, \cdot 2) & (1,0) & (1,0) & (\cdot 8, \cdot 1) \\ (0, \cdot 8) & (0,1) & (0,1) & (1,0) & (0,1) \\ (0, \cdot 7) & (\cdot 1, \cdot 8) & (0, \cdot 7) & (\cdot 9, \cdot 1) & (1,0) \end{bmatrix} \end{array}$$

Clearly the relation is intuitionistic fuzzy reflexive and intuitionistic fuzzy antisymmetric from its definition. Also, it is max-min, min-max transitive (see (3.7)). The following table describes the lub and glb for different pain of elements of *X*.

| Element pair | lub | glb |
|---|---|---|
| {a, b} | *b* | *a* |
| {a, c} | *a* | *c* |
| {a, d} | *d* | *a* |
| {a, e} | *e* | *a* |
| {b, c} | *b* | *c* |
| {b, d} | *d* | *b* |
| {b, e} | *b* | *e* |
| {c, d} | *d* | *c* |
| {c, e} | *e* | *c* |
| {d, e} | *d* | *e* |

So, the set *X* with intuitionistic fuzzy partial ordering *R* defined over it as above, is an intuitionistic fuzzy lattice.

**Theorem 3.1:** In an intuitionistic fuzzy lattice *(L, R)* for any two elements $a, b \in L$,

$$\mu_R(a,b) > 0 \text{ or } \{\mu_R(a,b) = 0 \text{ and } \nu_R(a,b) < 1\}$$
$$\Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$$

**Proof** We have

$$a \wedge b = a \Rightarrow \mu_{L(R,\{a,b\})}(a) > 0$$
$$\Rightarrow \mu_R(a,b) > 0 \text{ or} \{\mu_R(a,b) = 0 \text{ and } \nu_R(a,b) < 1\}.$$

Conversely,

$$\mu_R(a,b) > 0 \text{ or} \{\mu_R(a,b) = 0 \text{ and } \nu_R(a,b) < 1\}$$
$$\Rightarrow a \in R_{\leq[b]} \Rightarrow a \in L(R,\{a,b\}).$$

This together with $\mu_R(a,b) > 0$ implies that 'a' is the glb of *{a, b}* or $a \wedge b$ =a.

This completes the proof.

The following results which have been established in case of fuzzy lattices can be easily extended to the setting of intuitionistic fuzzy lattices as we have shown in proving the above theorem.

**Theorem 3.2:** Let (L, R) be an intuitionistic fuzzy lattice. Then the idempotent, commutative, associative and absorption properties for the operations $\wedge$ *and* $\vee$ hold.

**Theorem 3.3:** For all a, b, c $\in$ L, where (L, R) is an IF-lattice,

$$\mu_R(a,b) > 0 \text{ or } \{\mu_R(a,b) = 0 \text{ and } \nu_R(a,b) < 1\}$$
$$\Rightarrow \{\mu_R(a \wedge c, b \wedge c) > 0 \text{ and } \mu_R(a \vee c, b \vee c) > 0\}$$

or $$\{\mu_R(a \wedge c, b \wedge c) = 0, \mu_R(a \vee c, b \vee c) = 0, \nu_R(a \wedge c, b \wedge c) < 1, \nu_R(a \vee c, b \vee c) < 1\},$$

.

**Theorem 3.4:** For all a ,b ,c $\in$ L, where (L,R) is an IF-lattice,

(i) $$\{\mu_A(a,b) > 0 \text{ and } \mu_R(a,c) >. 0\} \text{ or } \{\mu_R(a,b) = 0, \nu_R(a,b) < 1, \mu_R(a,c) = 0, \nu_R(a,c) < 1\}$$

$$\Rightarrow \{\mu_R(a, b \vee c) > 0 \text{ and } \mu_R(a, b \wedge c) > 0\} \text{ or}$$
$$\{\mu_R(a, b \vee c) = 0, \mu_R(a, b \wedge c) = 0, \nu_R(a, b \vee c) < 1, \nu_R(a, b \wedge c) < 1\}$$

(ii) $$\{\mu_A(b,a) > 0 \text{ and } \mu_R(c,a) .< 0\} \text{ or } \{\mu_R(b,a) = 0, \nu_R(b,a) < 1, \mu_R(c,a) = 0, \nu_R(c,a) < 1\}$$

$$\Rightarrow \{\mu_R(b \wedge c, a) > 0, \mu_R(b \vee c, a) > 0\} \text{ or}$$

$$\begin{Bmatrix} \mu_R(b \wedge c, a) = 0, \nu_R(b \wedge c, a) < 1, \\ \mu_R(b \vee c, a) = 0, \nu_R(b \wedge c, a) < 1 \end{Bmatrix}$$

**Theorem 3.5:** For a, b, c $\in$ L, where (L, R) is an IF-lattice the distributive inequalities hold.

**Theorem 3.6:** For all a, b, c $\in$ L, where *L* is an IF-lattice,

$$\mu_R(a,c) > 0 \Leftrightarrow \mu_R(a \vee (b \wedge c), (a \vee b) \wedge c) > 0 \text{ and}$$

$$\{\mu_R(a,c) = 0, \nu_R(a,c) < 1\}$$

$$\Leftrightarrow \{\mu_R(a \vee (b \wedge c), (a \vee b) \wedge c) = a,$$

$$\nu_R(a \vee (b \wedge c), (a \vee b) \wedge c) < 1\}$$

**Definition 3.13:** An IF-lattice (L, R) is said to be complete if every nonempty subset of L has a lub and glb.

**Definition 3.14:** An IF-lattice (L, R) is said to be bounded if $\exists$ two elements, 0, 1 $\in$ L such that $\left[\mu_R(0,x) > 0 \text{ or} \{\mu_R(0,x) = 0 \text{ and } \nu_R(0,x) < 1\}\right]$ and $\left[\mu_R(x,1) > 0 \text{ or} \{\mu_R(x,1) = 0 \text{ and } \nu_R(x,1) < 1\}\right]$ for all x $\in$ L.

**Definition 3.15:** IF-lattice (L, R) is said to be distributive if and only if for all a, b, c $\in$ L,

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

and

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

**Definition 3.16:** An IF-lattice (L, R) is said to be modular if $a \vee (b \wedge c) = (a \vee b) \wedge c$ whenever $\mu_R(a,c) . > 0 \text{ or } \{\mu_R(a,c) = 0 \text{ and } \nu_R(a,c) < 1\}$ for all a, b, c $\in$ L.

**Definition 3.17:** Let (L, R) be a bounded IF-lattice and we denote the lower and upper bounds of L by 0 and 1 respectively. An element $a' \in L$ is said to be a complement of a $\in$ L if and only if $a \wedge a' = 0 \text{ and } a \vee a' = 1$.

The following theorems can be proved as in the corresponding crisp cases:

**Theorem 3.7:** Every distributive IF-lattice is modular.

**Theorem 3.8:** If *(L, R)* is a complemented distributive IF-lattice then the two De Morgan's Laws $(a \vee b)' = a' \wedge b'$ and $(a \wedge b)' = a' \vee b'$ hold for all *a, b ∈ L.*

**Example 3.1:** We define an intuitionistic fuzzy pentagonal lattice as follows: Let $L = \{0, a_1, a_2, a_3, 1\}$. The intuitionistic fuzzy partial ordering relation is defined in terms of the matrix given below. Here the entries denote the values of the membership and non-membership function as an ordered pair

| IFP | 0 | $a_1$ | $a_2$ | $a_3$ | 1 |
|---|---|---|---|---|---|
| **0** | (1,0) | >0 or (0,<1) | >0 or (0,<1) | >0 or (0, <1) | >0 or (0, <1) |
| $a_1$ | (0,1) | (1,0) | (0,1) | - | >0 or (0, <1) |
| $a_2$ | (0,1) | - | (1,0) | - | >0 or (0, <1) |
| $a_3$ | (0,1) | - | - | (1,0) | >0 or (0, <1) |
| **1** | (0,1) | (0,1) | (0,1) | (0,1) | (1,0) |

Here '>0' means any real number in (0, 1) can be assigned for the membership functional value and there is no restriction on the non-membership functional value.

Also, (0, <1) means the value of the membership function is zero and that of non-membership function in strictly less than 1.

Finally, '–' means that no value exists for these slots; that is the values are undefined.

**Example 3.2:** We define the intuitionistic fuzzy diamond lattice as follows: Let $L = \{0, b_1, b_2, b_3, 1\}$. The intuitionistic fuzzy partial ordering relation is defined in term of the matrix given below:

| IFD | 0 | $b_1$ | $b_2$ | $b_3$ | 1 |
|---|---|---|---|---|---|
| **0** | (1,0) | >0 or (0,<1) | >0 or (0,<1) | >0 or (0, <1) | >0 or (0, <1) |
| $b_1$ | (0,1) | (1,0) | - | - | >0 or (0, <1) |
| $b_2$ | (0,1) | - | (1,0) | - | >0 or (0, <1) |
| $b_3$ | (0,1) | - | - | (1,0) | >0 or (0, <1) |
| **1** | (0,1) | (0,1) | (0,1) | (0,1) | (1,0) |

The interpretations of the entries in the table have some meanings as in case of Example 3.1. In both those cases we get an infinite number of such lattices.

**Definition 3.18:** An intuitionistic fuzzy chain is an IF-partially ordered set (L, R) in which for two elements a,b$\in$ L, either $\mu_R(a,b) > 0$ *or* $\{\mu_R(a,b) = 0$ *and* $\nu_R(a,b) < 1\}$ *or*

$$\mu_R(b,a) > 0 \ \ or \{\mu_R(b,a) = 0 \ \ and \ \nu_R(b,a) < 1\}$$

We state the following two properties:

**Theorem 3.9:** Every intuitionistic fuzzy chain is a distributive IF-lattice.

**Theorem 3.10:** In a complemented distributed IF-lattice *(L, R)* $\forall \ a,b \in L$

$$\mu_R(a,b) > 0 \ or \ \{\mu_R(a,b) = 0, \nu_R(a,b) < 1\} \Leftrightarrow a \wedge b' = 0 \Leftrightarrow a \vee b' = 1$$

$$\Leftrightarrow \mu_R(b', a') > 0 \ or \ \{\mu_R(b', a') = 0, \ \nu_R(b', a') < 1\}.$$

## IV. Intuitionistic Fuzzy Boolean Algebra

In this section we define a special type of IF-lattice which is called IF-Boolean algebra and establish some of its properties.

**Definition 4.1:** A complemented distributive IF-lattice is called an *IF-Boolean algebra*.

Every complement IF-lattice is necessarily bounded. So, an IF-Boolean algebra is necessarily bounded. We denote it by $\underline{B} = (B\ ,\ \wedge\ ,\ \vee, 0, 1, )$, where *B* is a distributive bounded IF-lattice with bounds 0 and 1 and every element $a \in B$ has an unique complement denoted by $a'$.

**Definition 4.2:** Let $\underline{B} = (B, \wedge, \vee, 0, 1, ')$ be a IF-Boolean algebra. For any two elements a and b in *B*, we define the operation 'ring sum' denoted by $\oplus$ as

(4.1) $$a \oplus b = (a \wedge b') \vee (a' \wedge b).$$

'$\oplus$' is a well defined operation on $\underline{B}$.

The following properties of IF-Boolean algebra can be proved as in case of fuzzy Boolean algebra. We only state these results.

**Theorem 4.1:** Let $\underline{B}$ be an IF-Boolean algebra. Then

(4.2) $a \oplus b = (a \vee b) \wedge (a \wedge b)'$, $\forall a, b \in B$

(4.3) $a \oplus b = b \oplus a$, $\forall a, b \in B$

(4.4) $\oplus$ is associative

(4.5) $a \oplus 0 = 0 \oplus a = a$, $\forall a \in B$

(4.6) $a \oplus 1 = 1 \oplus a = a'$, $\forall a \in B$

(4.7) $a \oplus a = 0$, $\forall a \in B$

(4.8) $a \wedge (b \oplus c) = (a \wedge b) \oplus (a \wedge c)$, $\forall a, b, c \in B$

**Definition 4.3:** In any IF-Boolean algebra $\underline{B} = (B, \wedge, \vee, 0, 1, ')$ a ring product (1) is defined by

(4.9) $$a \odot b = a \wedge b,\ \forall a, b \in B$$

**Definition 4.4:** A complemented distributive IF-lattice B with the binary operations $\oplus$ *and* $\odot$ is a IF-Boolean ring with identity 1.

**Theorem 4.2:** In a IF-Boolean algebra $\underline{B}$,

(4.10) $a \oplus b = 0 \Leftrightarrow a = b$, $\forall a, b \in B$

(4.11) $a \vee (1 \oplus a) = 1$ and

(4.12) $a \odot (1 \oplus a) = 0$, for all $a \in B$

**Definition 4.5:** Let $(L, \wedge, \vee)$ be an IF-lattice with a lower bound 0. An immediate successor of 0 is called an *atom*.

Thus, $a \neq 0$ is an atom if $\{\mu_R(a, b) > 0\ and\ \mu_R(b, a) > 0\}$ or $\{\mu_R(a, b) = 0\ , \nu_R(a, b) < 1,\ \mu_R(b, a) = 0,\ \mu_R(b, a) < 1\} \Rightarrow b = 0$ or *b*=*a*, where *R* is the *IF* partially ordered relation on *L*.

**Definition 4.6:** Let $(L, \wedge, \vee)$ be on IF-lattice with a upper bound 1. An immediate predecessor of 1 is called on antiatom.

Thus $a \neq 1$ is an antiatom if $\mu_R(a, b) > 1\ and\ \mu_R(b, 1) > 0 \Rightarrow b = a\ or\ b = 1$.

**Definition 4.7:** Let $(L, \wedge, \vee)$ be an IF-lattice. An element $a \in L$ is said to be join irreducible if

(4.13) $$a = a_1 \vee a_2 \Rightarrow a = a_1\ or\ a = a_2$$

a is said to meet irreducible if

(4.14) $a = a_1 \wedge a_2 \Rightarrow a = a_1 \; or \; a = a_2$.

**Theorem 4.3:** Let $(B, \wedge, \vee, 0, 1, ')$ be a IF-Boolean algebra. Then the following hold:

(4.15) a is join irreducible $\Leftrightarrow a = a_1 \vee a_2$

$$\Rightarrow \mu_R(a_2, a_1) > 0 \text{ or } \mu_R(a_1, a_2) > 0 \text{ or}$$
$$\{\mu_R(a_2, a_1) = 0, \nu_R(a_2, a_1) < 1\}$$
$$or \; \{\mu_R(a_1, a_2) = 0, \nu_R(\nu_1, \nu_2) < 1\},$$

and

(4.16) a is meet irreducible $\Leftrightarrow a = a_1 \wedge a_2$

$$\Rightarrow \mu_R(a_2, a_1) > 0 \text{ or } \mu_R(a_1, a_2) > 0 \text{ or}$$
$$\{\mu_R(a_1, a_2) = , \nu_R(a_1, a_1) < 1\}$$
$$\text{or } \{\mu_R(a_2, a_1) = 0, \nu_R(a_2, a_1) < 1\}$$

**Theorem 4.4:** In any IF-lattice with a lower bound 0, every atom is join irreducible.

**Theorem 4.5:** Let $(B, \wedge, \vee, 0, 1, 1)$ be a IF-Boolean algebra. Then $0 \neq a \in B$ is an atom if and only if it is join irreducible.

## V. CONCLUSIONS

The notion of Intuitionistic fuzzy lattice introduced in this paper is an extension of the corresponding definition of fuzzy lattice introduced in [12]. The advantage in this definition is that we consider a partially ordered intuitionistic fuzzy relation is defined over a set, which provides a natural partial ordering instead of the earlier cases where an intuitionistic fuzzy set is taken and a normal partially ordered relation is defined. Many concepts related to this notion are defined and properties are established. Some of these properties have been proved and the others can be proved in a way similar to the fuzzy case. Two special IF-lattices have been defined. Another special case and perhaps the most important one that of IF-Boolean algebra is defined and its properties are established. Since the intuitionistic fuzzy lattices are more realistic than the fuzzy lattices the results established in this paper will cover wider area of applications. The notion of intuitionistic fuzzy Boolean algebra will lead to the study of intuitionistic fuzzy Boolean expressions and possibly the notion of intuitionistic fuzzy gates can be developed.

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