On a M/G/1 Queue Subject to Two Types of Failures with General or Deterministic Repair Times and Two Types of Optional Server Vacations with General or Deterministic Vacation Times(M/G/$\binom{G}{D}$Repair Times/$\binom{G}{D}$Vacation Times/1 Queue)

Kailash C. Madan
Department of Mathematical Sciences, Ahlia University, Bahrain
kmadan@ahlia.edu.bh; kcmadan@yahoo.com

Abstract—**We study an M/G/1 queueing system subject to random failures of two types. Type 1 failure needs repairs having general repair time distribution and type 2 failure needs deterministic repair time of fixed length. In addition to two types of failures, we assume that after completion of every service the server has the option to take a vacation with probability δ or to keep staying in the system with probability $1-\delta$. When the server opts for a vacation, he may take a type 1 vacation with general vacation time or a type 2 vacation with a deterministic vacation time.**

Keywords-Poisson arrivals, random failures, general repair times, deterministic repair times, optional vacations, general vacation times, deterministic vacation times, steady state

I. INTRODUCTION

Many researchers on queueing theoryhave studied queues with different types of service interruptions including random breakdowns or server vacations with several different vacation policies. This paper focuses on a single server queue with Poisson arrivals, General service subject to two types of random failures with heterogeneous repair time distributions, general or deterministic. After completion of each service, the server may take optional server vacations of two types with heterogeneous vacation time distributions, general or deterministic. For some earlier papers on breakdowns, we refer the reader to Gaver [3], Thiruengadam [12], Avi-Izhac and Naor [1] and Madan [4] and for queueing papers for vacations, we mention the works of a few authors such as Doshi [2], Takagi [10], Madan [5] and Tian and Zhang [11]. Further, a few more recent papers on queues dealing with a mixture of both aspects of breakdowns and vacations are Maraghi, F. et al [7], Khalaf, R, et al [9], Monita et al [8] and Madan [6]. In this paper, we investigate an M/G/1 queue with two types of system failures. We assume that type 1 failure, which may occur with probability$\propto_1$, requires general repair time distribution and type 2 failure, which may occur with probability$\propto_2$, $\propto_1+\propto_2=1$, requires a deterministic repair time of constant length. In addition, we assume that after completion of a service the server may take a vacation with probability δ or may continue staying in the system with probability $1-\delta$. When the server opts for a vacation, then with probability $p_1$he may take a type 1 vacation with general vacation time or with probability p_2 he may take a type 2 vacation with a deterministic vacation time.

II. DESCRIPTION OF THE MATHEMATICAL MODEL

- We assume single Poisson arrivals with mean arrival rate $\boldsymbol{\omega}$ (>0)
- The service time '*S*' follows a general distribution. Let$\boldsymbol{B}(\boldsymbol{x})and\boldsymbol{b}(\boldsymbol{x})$respectively be the distribution function and the density function of the service time *S* and let $\mu(x)dx$ be the conditional probability of completion of service, given that the elapsed time is *x* , so that

  $\mu(x)=\frac{b(x)}{1-B(x)}$, and, therefore, b(x)= $\mu(x)e^{-\int_0^x \mu(t)dt}$ (1)

- The server is subject to two types of random failures. Let $\propto_1\ dt$be the probability that a breakdown will occur due to a type 1 failure during the short time interval $(t, t + dt]$ and let$\propto_2\ dt$be the probability that a breakdown will occur due to a type 2 failure during the short time interval $(t, t + dt]$
- We assume that a breakdown can occur only when the server is proving service and not when it is idle.
- We assume that the customer whose service is interrupted because of a sudden failure returns at the head of the queue and waits until the repairs of the server are complete.
- We assume that as the result of a breakdown, the server undergoes repair process immediately after the occurrence of any type of failure.
- When a breakdown of type 1 occurs, the repair time 'R_1' follows a general distribution. Let $H(x)$ and $h(x)$ respectively be the distribution function and the density function of the repair time 'R_1' and let $\theta(x)dx$ be the conditional probability of completion of a repair of type 1 failure, given that the elapsed time is x , so that

  $\theta = \frac{h(x)}{H(x)}$, and, therefore, $\theta(x) = h(x)e^{-\int_0^x \theta(t)dt}$ (2)
- When a breakdown of type 2 occurs, the repair time follows a deterministic repair time of constant length π.
- After completion of each service, the server has a choice of taking one of the two types of vacations with probability δ , or with probability $1-\delta$, the server may continue staying in the system. We further assume that the server has the choice of taking a type 1 vacation of random length following a general distribution withprobability$p_1$or a type 2 vacation of constant duration d with probability p_2, $p_1 + p_2 = 1$.
- Let $G(x)$ and $g(x)$ respectively be the distribution function and the density function of the vacation time V and let $\vartheta(x)dx$ be the conditional probability of completion of a vacation, given that the elapsed time is x , so that
- $\vartheta(x) = \frac{g(x)}{1-G(x)}$ and, therefore, $\vartheta(x) = g(x)e^{-\int_0^x \vartheta(t)\,dt}$ (3)
- We assume that on completion of either type of repair or completion of either type of vacation, the server instantly takes up a customer (at the head of the queue) for service if there is a customer waiting in the queue. However, if on returning the server finds the queue empty, the server still joins the system and remains idle until a new batch of customers arrives in the system.
- Various stochastic processes involved in the system are independent of each other.

III. DEFINITIONS AND NOTATIONS

We define the following probabilities:

$W_n(x, t)$: probability that at time *t* there are n (≥ 0) customers in the queue excluding one customer in service with elapsed service time *x*. Accordingly, $W_n(t) = \int_{n=0}^{\infty} W_n(x, t)\,dx$ denotes the probability that at time t there are $n \geq 0$ customers in the queue excluding one customer in service irrespective of the value of x.

$R_n^{(1)}(x, t)$: Probability that at time t there are $n \geq 1$ customers in the queue and the server is under repairs for a type 1 failure with elapsed repair time x. Accordingly, $R_n^{(1)}(t) = \int_0^{\infty} R_n^{(1)}(x, t)dx$ denotes the probability that at time t there are $n \geq 1$ customers in the queue and the server is under repairs irrespective of the value of x.

$R_n^{(2)}(t)$: Probability that at time t there are $n \geq 1$ customers in the queue and the server is under repairs for a type 2 failure

$V_n^{(1)}(x, t)$: probability that at time *t* there are $n(> 0)$ customers in the queue and the server is on type 1 vacation with elapsed vacation time x. Accordingly, $V_n^{(1)}(t) := \int_0^{\infty} V_n^{(1)}(x, t)dx$ denotes the probability that at time t there are $n(> 0)$ customers in the queue irrespective of the state of the system.

$V_n^{(2)}(t)$: Probability that at time t, the server is on type 2 vacation with deterministic vacation time.

$P_n(t) = W_n(t) + R_n^{(1)}(t) + R_n^{(2)}(t) + V_n^{(1)}(t) + V_n^{(2)}(t)$: Probability that at time *t* there are n (≥ 0) customers in the queue irrespective of the state of the system.

Q (t): probability that there is no customer in the system and the server is idle.

Now, if the steady sate exists, we define the following limiting probabilities as the steady state probabilities corresponding to the probabilities defined above for the various states of the system:

$\mathrm{Lim}_{t\to\infty} W_n(x,t) = W_n(x)$, $\mathrm{Lim}_{t\to\infty} W_n(t) = W_n$, $\mathrm{Lim}_{t\to\infty} R_n^{(1)}(x,t) = R_n^{(1)}(x)$, $\mathrm{Lim}_{t\to\infty} R_n^{(1)}(t) = R_n^{(1)}$,
$\mathrm{Lim}_{t\to\infty} V_n^{(1)}(x,t) = V_n^{(1)}(x)$, $\mathrm{Lim}_{t\to\infty} V_n^{(1)}(t) = V_n^{(1)}$, $\mathrm{Lim}_{t\to\infty} V_n^{(2)}(t) = V_n^{(2)}$
$\mathrm{Lim}_{t\to\infty} R_n^{(2)}(t) = R_n^{(2)}$ (4)

We further assume that K_r is the probability of r arrivals during the repair time of type 2 failure and L_r the probability of r arrivals during the period of type 2 vacation

$K_r = \frac{\exp(\omega d)(\omega d)^r}{r\,!}$, $L_r = \frac{\exp(\omega\pi)(\omega\pi)^r}{r\,!}$, $r = 0, 1, 2, \ . \ . \ .$ (5)

Next, we define the following Probability Generating Functions (PGFs):

$W(x,z) = \sum_{n=0}^{\infty} W_n(x) z^n,\ W(z) = \sum_{n=0}^{\infty} W_n z^n,$ (6)

$R^{(1)}(x,z) = \sum_{n=1}^{\infty} R_n^{(1)}(x) z^n,\ R^{(1)}(z) = \sum_{n=1}^{\infty} R_n^{(1)} z^n,$ (7)

$V^{(1)}(x,z) = \sum_{n=0}^{\infty} V_n^{(1)}(x) z^n,\ V^{(1)}(z) = \sum_{n=0}^{\infty} V_n^{(1)} z^n,$ (8)

$V^{(2)}(z) = \sum_{n=0}^{\infty} V_n^{(2)} z^n\ , R^{(2)}(z) = \sum_{n=0}^{\infty} R_n^{(2)} z^n$ (9)

$K(z) = \sum_{n=0}^{\infty} K_n z^n = \sum_{n=0}^{\infty} \frac{e^{-\omega d}(\omega d)^n}{n!} = e^{-\omega d(1-z)}, |z\} < 1$ (10)

$L(z) = \sum_{n=0}^{\infty} L_n z^n = \sum_{n=0}^{\infty} \frac{e^{-\omega\pi}(\omega\pi)^n}{n!} = e^{-\omega\pi(1-z)}, |z\} < 1.$ (11)

IV. STEADY STATE EQUATIONS

Applying the usual probability arguments based on the underlying assumptions, we obtain the following steady state equations:

$\frac{d}{dx} W_n(x) + (\boldsymbol{\omega} + \mu(x) + \propto_1 + \propto_2) W_n(x) = \boldsymbol{\omega} W_{n-1}(x), n \geq 1,$ (12)

$\frac{d}{dx} W_0(x) + (\boldsymbol{\omega} + \mu(x) + \propto_1 + \propto_2) W_0(x) = 0,$ (13)

$\frac{d}{dx} R_n(x) + (\omega + \theta(x)) R_n(x) = \omega R_{n-1}(x), n \geq 2,$ (14)

$\frac{d}{dx} R_1(x) + (\omega + \theta(x)) R_1(x) = 0,$ (15)

$\frac{d}{dx} V_n^{(1)}(x) + (\boldsymbol{\omega} + \vartheta(x)) V_n^{(1)}(x)(x) = \boldsymbol{\omega} V_{n-1}^{(1)}(x)(x), n \geq 1,$ (16)

$\frac{d}{dx} V_0^{(1)}(x) + (\boldsymbol{\omega} + \vartheta(x)) W_0(x) = 0,$ (17)

$V_n^{(2)} = \delta p_2 \int_0^{\infty} W_n(x)\mu(x)dx, n \geq 0,$ (18)

$R_{n+1}^{(2)} = \propto_2 W_n$ (19)

$\omega\, Q = (1-\delta)\int_0^{\infty} W_0(x)\mu(x)dx + \int_0^{\infty} V_0^{(1)}(x)\vartheta(x)dx + V_0^{(2)} k_0.$

(20)

The above equations would be solved subject to the following boundary conditions:

$$W_n(0) = (1-\delta)\int_0^{\infty} W_{n+1}(x)\mu(x) + \int_0^{x} V_{n+1}^{(1)}(x)\vartheta(x)dx$$
$$+ \int_0^{\infty} R^{(1)}{}_{n+1}(x)\theta(x)dx,$$
$$+(V_1^{(2)}k_n + V_2^{(2)}k_{n-1} + \ . \ . \ . , + V_{n+1}^{(2)}k_0) + (R_1^{(2)}l_n$$
$$+ R_2^{(2)}l_{n-1} + \ . \ . \ . , + R_{n+1}^{(2)}l_0)\ , n \geq 1,$$

(21)

$$W_0(0) = (1-\delta)\int_0^{\infty} W_1(x)\mu(x) + \int_0^{x} V_1^{(1)}(x)\vartheta(x)dx$$
$$+ \int_0^{\infty} R_1(x)\theta(x)dx,$$
$$+(V_1^{(2)}k_0 + V_0^{(2)}k_1 + (R_1^{(2)}l_0 + R_0^{(2)}l_1), \quad (22)$$

$R^{(1)}{}_{n+1}(0) = \propto_1 W_n, n \geq 0,$ (23)

$V_n^{(1)}(0) = \delta\, p_1 \int_0^{\infty} W_n(x)\mu(x)dx, n \geq 0\ .$ (24)

V. STEADY STATE SOLUTION

We multiply both sides of equation (12) by suitable powers of z, add equation (13) to the result and use (6) and simplify. Thus, we get

$$\frac{d}{dz} W(x,z) + (\omega - \omega z + \mu(x) + \propto_1 + \propto_2) W(x,z) = 0. \quad (25)$$

Similarly, from (14) and (15) we get

$$\frac{d}{dz} R(x,z) + \big(\omega - \omega z + \theta(x)\big) R(x,z) = 0. \quad (26)$$

And from (16) and (17) we obtain

$$\frac{d}{dz} V^{(1)}(x,z) + \big(\omega - \omega z + \vartheta(x)\big) V^{(1)}(x,z) = 0. \quad (27)$$

And with similar operation on (18) and (19) separately we obtain

$$V^{(2)}(z) = \delta p_2 \int_0^\infty W(x,z)\mu(x)dx. \quad (28)$$

$$R^{(2)}(z) = \propto_2 zW(z) \quad (29)$$

Yet again, we use similar operations on the boundary conditions (21) - (24), use (20) and simplify. We thus obtain

$$zW(0,z) = (1-\delta)\int_0^\infty W(x,z)\mu(x) + \int_0^x V^{(1)}(x,z)\vartheta(x)dx + \int_0^\infty R(x,z)\theta(x)dx$$
$$+ V^{(2)}(z)K(z) + R^{(2)}(z)L(z) + \omega(z-1)Q, \quad (30)$$

$$R^{(1)}(0,z) = \propto_1 zW(z), \quad (31)$$

$$V^{(1)}(0,z) = \delta p_1 \int_0^\infty W(x,z)\mu(x)dx\ . \quad (32)$$

Now, we integrate equations (25), (26) and (27) between the limits 0 and x. Thus, we obtain

$$W(x,z) = W(0,z)e^{-(\omega-\omega z+\propto_1+\propto_2)-\int_0^x \mu(t)dt}, \quad (33)$$

$$R^{(1)}(x,z) = R^{(1)}(0,z)e^{-(\omega-\omega z)-\int_0^x \theta(t)dt}, \quad (34)$$

$$V^{(1)}(x,z) = V^{(1)}(0,z)e^{-(\omega-\omega z)-\int_0^x \vartheta(t)dt}, \quad (35)$$

where $W(0,z)$, $R(0,z)$ and $V^{(1)}(0,z)$ are given by (30), (31) and (32) respectively.

Again integrating (33), (34) and (35) with respect to x we obtain

$$W(z) = W(0,z)\left(\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right), \quad (36)$$

$$R^{(1)}(z) = R^{(1)}(0,z)\left(\frac{1-\bar{H}(\omega-\omega z)}{\omega-\omega z}\right), \quad (37)$$

$$V^{(1)}(z) = V^{(1)}(0,z)\left(\frac{1-\bar{G}(\omega-\omega z)}{\omega-\omega z}\right), \quad (38)$$

Where $\bar{B}(\omega - \omega z + \propto) = \int_0^\infty e^{-(\omega-\omega z+\propto_1+\propto_2)x} dB(x)$ is the Laplace-Stieltjes transform of the service time S, $\bar{H}(\omega - \omega z) = \int_0^\infty e^{-(\omega-\omega z)x} dH(x)$ is the Laplace-Stieltjes transform of the repair time R and $\bar{G}(\omega - \omega z) = \int_0^\infty e^{-(\omega-\omega z)x} dG(x)$ is the Laplace-Stieltjes transform of the vacation time V.

Next, we multiply equations (32), (33) and (34) by $\mu(x)$, $\theta(x)$ and $\vartheta(x)$ respectively and get

$$\int_0^\infty W(x,z)\,\mu(x)dx = W(0,z)\bar{B}\,(\omega - \omega z + \propto)\ , \quad (39)$$

$$\int_0^\infty R(x,z)\,\theta(x)dx = R(0,z)\overline{H}\,(\omega - \omega z)\ , \quad (40)$$

$$\int_0^\infty V^{(1)}(x,z)\,\vartheta(x)dx = V^{(1)}(0,z)\overline{G}\,(\omega - \omega z)\ . \quad (41)$$

Now, using equations (39), (40) and (41) in (28), (30), (31) and (32), simplifying and further using (36), (37) and (38), we obtain

$$W(z) = \frac{\omega(z-1)\left(\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right)Q}{z-[(1-\delta)+\delta p_1\bar{G}(\omega-\omega z)+\delta p_2 K(z)]\bar{B}(\omega-\omega z+\propto_1+\propto_2) - [\propto_1 z\bar{H}(\omega-\omega z)+\propto_2 z\,L(z)]\left[\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right]} \quad (42)$$

$$R^{(1)}(z) = \frac{\propto_1 z\left(\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right)(\bar{H}(\omega-\omega z)-1)\,Q}{z-[(1-\delta)+\delta p_1\bar{G}(\omega-\omega z)+\delta p_2 K(z)]\bar{B}(\omega-\omega z+\propto_1+\propto_2) - [\propto_1 z\bar{H}(\omega-\omega z)+\propto_2 z\,L(z)]\left[\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right]}, \quad (43)$$

$$R^{(2)}(z) = \frac{\propto_2 z\, \omega(z-1)\left(\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right)Q}{z-[(1-\delta)+\delta p_1\bar{G}(\omega-\omega z)+\delta p_2 K(z)]\bar{B}(\omega-\omega z+\propto_1+\propto_2) - [\propto_1 z\bar{H}(\omega-\omega z)+\propto_2 z\, L(z)]\left[\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right]} \quad (44)$$

$$V^{(1)}(z) = \frac{\delta p_1\bar{B}(\omega-\omega z+\propto_1+\propto_2)(\bar{G}(\omega-\omega z)-1)Q}{z-[(1-\delta)+\delta p_1\bar{G}(\omega-\omega z)+\delta p_2 K(z)]\bar{B}(\omega-\omega z+\propto_1+\propto_2) - [\propto_1 z\bar{H}(\omega-\omega z)+\propto_2 z\, L(z)]\left[\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right]}, \quad (45)$$

$$V^{(2)}(z) = \frac{\delta p_2\, \bar{B}(\omega-\omega z+\propto_1+\propto_2)\, \omega(z-1)Q}{z-[(1-\delta)+\delta p_1\bar{G}(\omega-\omega z)+\delta p_2 K(z)]\bar{B}(\omega-\omega z+\propto_1+\propto_2) - [\propto_1 z\bar{H}(\omega-\omega z)+\propto_2 z\, L(z)]\left[\frac{1-\bar{B}(\omega-\omega z+\propto_1+\propto_2)}{\omega-\omega z+\propto_1+\propto_2}\right]}. \quad (46)$$

Now, in order to determine the only unknown Q, we proceed as follows:

$$W(1) = \lim_{z\to 1} W(z) = \frac{\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)Q}{1-[\delta p_1\omega E(V)+\delta p_2\omega d]\bar{B}(\propto_1+\propto_2) - [\omega+\propto_1+\propto_2+\propto_1\omega E(R)+\propto_2\omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, (47)$$

$$R^{(1)} = \lim_{z\to 1} R^{(1)}(z) = \frac{\propto_1\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)E(R)Q}{1-[\delta p_1\omega E(V)+\delta p_2\omega d]\bar{B}(\propto_1+\propto_2) - [\omega+\propto_1+\propto_2+\propto_1\omega E(R)+\propto_2\omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, (48)$$

$$R^{(2)}(1) = \lim_{z\to 1} R^{(2)}(z) = \frac{\propto_2\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)Q}{1-[\delta p_1\omega E(V)+\delta p_2\omega d]\bar{B}(\propto_1+\propto_2) - [\omega+\propto_1+\propto_2+\propto_1\omega E(R)+\propto_2\omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, \quad (49)$$

$$V^{(1)}(1) = \lim_{z\to 1} V^{(1)}(z) = \frac{\delta\, p_1\, \omega\, \bar{B}(\propto_1+\propto_2)\, E(V)Q}{1-[\delta p_1\omega E(V)+\delta p_2\omega d]\bar{B}(\propto_1+\propto_2) - [\omega+\propto_1+\propto_2+\propto_1\omega E(R)+\propto_2\omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, \quad (50)$$

$$V^{(2)}(1) = \lim_{z\to 1} V^{(2)}(z) = \frac{\delta\, p_2\, \omega\, \bar{B}(\propto_1+\propto_2)\, Q}{1-[\delta p_1\omega E(V)+\delta p_2\omega d]\bar{B}(\propto_1+\propto_2) - [\omega+\propto_1+\propto_2+\propto_1\omega E(R)+\propto_2\omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, \quad (51)$$

Where E(R) is the average repair time and E(V) is the average vacation time.

Next, we use the results found in (5.21), (5.22), (5.23) and (5.24) in the normalizing condition:

$$Q + W(1) + R^{(1)}(1) + R^{(2)}(1) + V^{(1)}(1) + V^{(2)}(1) = 1. \quad (52)$$

On simplifying, (52) yields

$$Q = \frac{1-[\delta\, p_1\ \omega E(V)+\delta p_2\omega d]\bar{B}(\propto_1+\propto_2)-[\omega+\propto_1+\propto_2+\propto_1\omega E(R)+\propto_2\omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}{1+\delta p_2\omega(1-d)\bar{B}(\propto_1+\propto_2)+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}. \quad (53)$$

Note that the result (53) gives the probability that the server is idle and the stability condition that emerges from this result is given by

$$[\delta\, p_1\ \omega E(V) + \delta p_2\omega d]\bar{B}(\propto_1+\propto_2) - [\omega +\propto_1+\propto_2+\propto_1\ \omega E(R) +\propto_2\ \omega\pi]\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right) < 1 \quad (54)$$

Next, on substituting the value of Q from (53) into equations (47) to (51), we obtain

$$W(1) = \frac{\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}{1+\delta p_2\omega(1-d)\bar{B}(\propto_1+\propto_2)+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, \quad (55)$$

This is the steady stae probability that the sever is busy providing service to customers.

$$R^{(1)}(1) = \frac{\propto_1\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)E(R)}{1+\delta p_2\omega(1-d)\bar{B}(\propto_1+\propto_2)+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, \quad (56)$$

This is the steady state probability that the server is under type 1 repairs.

$$R^{(2)}(1) = \frac{\propto_2\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}{1+\delta p_2\omega(1-d)\bar{B}(\propto_1+\propto_2)+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}, \quad (57)$$

This is the steady state probability that the server is under type 2 repairs.

$$V^{(1)}(1) = \frac{\delta\, p_1\, \omega\, \bar{B}(\propto_1 + \propto_2)\, E(V)}{1 + \delta p_2 \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (58)$$

This is the steady state probability that the server is on type 1 vacation.

$$V^{(2)}(1) = \frac{\delta\, p_2\, \omega\, \bar{B}(\propto_1 + \propto_2)}{1 + \delta p_2 \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (59)$$

This is the steady state probability that the server is on type 2 vacation.

VI. PARTICULAR CASES

Case 1: Breakdowns and Only Type 1 Server Vacations

In this case, we substitute $p_1 = 1$ and $p_2 = 0$ in the main results found in (53) to (59) and obtain

$$Q = \frac{1 - [\delta\ \omega E(V)] \bar{B}(\propto_1 + \propto_2) - [\omega + \propto_1 + \propto_2 + \propto_1 \omega E(R) + \propto_2 \omega \pi]\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}. \quad (60)$$

$$[\delta \omega E(V)] \bar{B}(\propto_1 + \propto_2) - [\omega + \propto_1 + \propto_2 + \propto_1\ \omega E(R) + \propto_2\ \omega \pi]\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right) < 1, \quad (61)$$

$$W(1) = \frac{\omega\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (62)$$

$$R^{(1)}(1) = \frac{\propto_1 \omega\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right) E(R}{1 + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (63)$$

$$R^{(2)}(1) = \frac{\propto_2 \omega\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (64)$$

$$V^{(1)}(1) = \frac{\delta\ \omega\ \bar{B}(\propto_1 + \propto_2)\ E(V)}{1 + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (65)$$

$$V^{(2)}(1) = 0, \quad (66)$$

Case 2: Breakdowns and Only Type 2 Vacations

In this case, we substitute $p_1 = 0$ and $p_2 = 1$ in the main results found in (53) to (59) and obtain

$$Q = \frac{1 - [\delta \omega d] \bar{B}(\propto_1 + \propto_2) - [\omega + \propto_1 + \propto_2 + \propto_1 \omega E(R) + \propto_2 \omega \pi]\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \delta \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}. \quad (67)$$

$$[\delta \omega d] \bar{B}(\propto_1 + \propto_2) - [\omega + \propto_1 + \propto_2 + \propto_1\ \omega E(R) + \propto_2\ \omega \pi]\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right) < 1, \quad (68)$$

$$W(1) = \frac{\omega\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (69)$$

$$R^{(1)}(1) = \frac{\propto_1 \omega\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right) E(R}{1 + \delta \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (70)$$

$$R^{(2)}(1) = \frac{\propto_2 \omega\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \delta \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (71)$$

$$V^{(1)}(1) = 0 \quad (72)$$

$$V^{(2)}(1) = \frac{\delta\ \omega\ \bar{B}(\propto_1 + \propto_2)}{1 + \delta \omega (1-d) \bar{B}(\propto_1 + \propto_2) + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}, \quad (73)$$

Case 3: Breakdowns and No Vacations

In this case, we substitute $\delta = 0$ in the main results found in (48) to (53) and obtain

$$Q = \frac{1 - [\omega + \propto_1 + \propto_2 + \propto_1 \omega E(R) + \propto_2 \omega \pi]\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}{1 + \propto_2 (1-\pi)\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right)}. \quad (74)$$

$$[\omega + \propto_1 + \propto_2 + \propto_1\ \omega E(R) + \propto_2\ \omega \pi]\left(\frac{1-\bar{B}(\propto_1 + \propto_2)}{\propto_1 + \propto_2}\right) < 1, \quad (75)$$

$W(1) = \frac{\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}{1+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)},$ (76)

$R^{(1)}(1) = \frac{\propto_1\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)E(R}{1+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)},$ (77)

$R^{(2)}(1) = \frac{\propto_2\omega\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)}{1+\propto_2(1-\pi)\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)},$ (78)

$V^{(1)}(1) = 0$(79)

$V^{(2)}(1) = 0$(80)

Case 4: No Breakdowns and Only Type 1 Vacations

In this case, we substitute $\propto_1 = 0 = \propto_2$ in the results of case 1. Consequently, we substitute $\bar{B}(0) = 1$and $\lim_{\propto_1\to 0,\propto_2\to 0}\left(\frac{1-\bar{B}(\propto_1+\propto_2)}{\propto_1+\propto_2}\right)$ = E(S), where E(S) is the mean service time. Thus, we get

$Q = 1 - [\delta\ \omega E(V)] - \omega E(S)$ (81)

$\omega\ \delta[\ E(V) + E(S)] < 1$ (82)

$W(1) = \omega E(S)$ (83)

$R^{(1)}(1) = 0$ (84)

$R^{(2)}(1) = 0$ (85)

$V^{(1)}(1) = \delta\omega E(V)$(86)

$V^{(2)}(1) = 0$(87)

Case 5: No Breakdowns and Only Type 2 Vacations

In this case, we substitute $\propto_1 = 0 = \propto_2$ in the results of case 2 and get

$Q = \frac{1-\delta\ \omega\ d-\omega\ E(S)}{1+\delta\ \omega(1-\ d)},$ (88)

$\delta\ \omega\ d + \omega\ E(S) < 1$,(89)

$W(1) = \frac{\omega\ E(S)}{1+\delta\ \ \omega\ (1-\ d)},$ (90)

$R^{(1)}(1) = 0\ ,$ (91)

$R^{(2)}(1)(1) = 0\ ,$ (92)

$V^{(1)}(1) = 0\ ,$ (93)

$V^{(2)}(1) = \frac{\delta\ \omega}{1+\delta\ \ \omega(1-d)}$.(94)

Case 6: No Breakdowns and No vacations

In this case, we substitute $\propto = 0$ in the results of case 3 and get

$Q = 1 - \omega\ E(S),$ (95)

$\omega E(S) < 1\ <\ 1.$ (96)

$W(1) =\ \omega\ E(S),$ (97)

$R^{(1)} = 0\ ,$ (98)

$R^{(2)}(1) = 0\ ,$ (99)

$V^{(1)}(1) = 0,$ (100)

$V^{(2)}(1) = 0.$ (101)

VII. CONCLUSION

In this paper we studied a new model of a queueing system which is subject to random breakdowns as well as server vacations. The new significant assumptions in the paper are that the server is subject to two types of failures. Type 1 failures need deterministic repair time and type 2 failures need general repair times. In addition, the server can take a type 1 vacation of constant length or a general vacation of variable length.

We have found the important meaningful steady state results in terms of probability generating functions in equations (42) to (53), the important steady state condition under which the steady state exists in equation (54) and the steady state probabilities of various states of the system in equations (55) to (59). Many meaningful and interesting results are derived cases 1 to 6. Essentially, these new results will be a good addition to the literature of queueing theory.

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REFERENCES

[1] B. Avi-Itzhak, and P. Naor, "Some queueing problems with the service station subject to breakdowns", Operations Research, 11, pp. 303 – 319, 1963.
[2] B. Doshi, "Queueing systems with vacations-a survey", Queueing Systems, 1, 29-66, 1986.
[3] D. P. Gaver, "A waiting line with interrupted service including priorities", Journal of Royal Statistical Society, Ser. - B,24, 73-90, 1962.
[4] Kailash C. Madan, "A queueing system with service interruptions", RevueRoumaine de MathematiquesPure et Appliques, Tome XVII, No.5, pp. 741-755, 1972.
[5] Kailash C. Madan, "An M/G/1 queue with optional deterministic server vacations", Metron, LVII No. 3-4, pp. 83-95, 1999.
[6] Kailash C. Madan, "A batch arrival single server queue with generalized Coxian-2 service and optional generalized Coxian-2vacation", Mathematical Sciences and Applications E Notes, 3-2, pp. 33-44, 2015.
[7] F. Maraghi, Kailash C. Madan andDarby-Dowman, "Batch arrival queueing system with random breakdowns and Bernoulli schedule server vacations having general vacation time distribution", International journal of Information and Management Sciences,1, pp. 55-70, 2009.
[8] Monita, Barua., Kailash C. Madan and Tilal, E.. "A two-stage batch arrival queue with reneging during vacation and Breakdown Periods", American Journal of Operations Research, 3, 570-580, 2013.
[9] F. Khalaf, Kailash C. and A. Lukas Cormac, "On a batch arrival queueing system equipped with a stand-by server during vacation periods or repair times of the main server, Journal of Probability and Statistics, Vol. 2011, Article ID 812726, pp. 1-11, 2011.
[10] H. Takagi, "Time dependent process of M/G/1 vacation models with exhaustive service", J. Appl. Prob., 29, 418-429, 1992.
[11] Tian, N. and Zhang, G.: Vacation queueing models - theory and applications, Springer-Verlag, New York, 2006.
[12] K. Thiruengadam, "Queueing with Breakdowns", Operations Research 11, 62-71, 1963.

AUTHOR PROFILE

Kailash C. Madan is a Professor of Mathematics and Statistics in the Department of Mathematical Sciences, College of Arts and Science of Ahlia University, Bahrain since 01 September 2003. He has earlier taught at Yarmouk University, Jordan for 7 years and the University of Bahrain for 16 years. A former Dean of the College of Mathematical Sciences and Information Technology, he is currently Advisor to the University President for Measurement and Evaluation.

Professor Madan has been a member of many scientific societies and has many research papers published in reputed international journals. He has attended numerous international conferences, presented papers and chaired many sessions. He has successfully supervised 3 Ph. D. students at Ahlia University under the joint program run by Ahlia University in Collaboration with Brunel University, London.