Absolute Summability Factor $|N, p_n|_k$ of Improper Integrals

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***Abstract*— In this paper, we defined the summability for integrals and established a theorem on absolute Nörlund summability $|N, p_n|_k$ factors of improper integral under sufficient conditions. Some auxiliary results (well-known) have also been deduced from the main results under suitable conditions.**

Keyword - Absolute Summability, Nörlund summability, Improper Integrals, Inequalities for Integrals.

I. INTRODUCTION

1. Summability factor concerning infinite series: Let $\sum_{n=0}^{\infty} a_n$ be an infinite series with sequence of partial sums, $s_n = \sum_{n=0}^{n} a_n$ and σ_n be the n^{th} Cesàro means of the series, i.e.,

$$\sigma_n = \frac{1}{n}\sum_{k=0}^{\infty} s_k \tag{1}$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be $|C, 1|_k$, $k \geq 1$ summable [5], if

$$\lim_{n\to\infty} \sigma_n = s, \tag{2}$$

and

$$\sum_{n=1}^{\infty} n^{k-1} |\sigma_n - \sigma_{n-1}|^k < \infty \tag{3}$$

2. Summability factor concerning improper integrals: Let f be a real valued continuous function of t in the interval $[0,\infty)$ and $s(x) = \int_0^x f(t)dt$. Then, $\int_0^\infty f(t)dt$ is said to be summable $|C, 1|$, if

$$\int_0^\infty |\sigma'(x)|\, dx < \infty. \tag{4}$$

and $\int_0^\infty f(t)dt$ is said to be summable $|C, 1|_k$, $k \geq 1$, if

$$\int_0^\infty x^{k-1} |\sigma'(x)|^k \, dx < \infty. \tag{5}$$

where $\sigma(x)$ is Cesàro mean of $s(x)$ and given by

$$\sigma(x) = \frac{1}{x}\int_0^x (x-t) f(t)dt, \tag{6}$$

or

$$\sigma(x) = \frac{1}{x}\int_0^x s(t)dt. \tag{7}$$

The Kronecker identity: $s(x) - \sigma(x) = v(x)$, where

$$v(x) = \frac{1}{x}\int_0^x tf(t)dt. \tag{8}$$

The condition (5) can be written as

$$\int_0^\infty \frac{1}{x} \mid v(x) \mid^k \, dx < \infty. \tag{9}$$

Considering the (*N, P_n*) and (*K,* 1*, α*) summability, Parashar [9] obtained the minimum set of conditions for an infinite series to be (*K,* 1*, α*) summable. In 1986, Bor [1] found the relationship between two summability techniques $|C, 1|_k$ and $|\bar{N}, p_n|_k$ and in [2], he used the $|\bar{N}, p_n|_k$ for generalization of a theorem based on minimal set of sufficient conditions for infinite series. In 2016, Sonker and Munjal [10] determined a theorem on generalized absolute Cesàro summability with the sufficient conditions for infinite series and in [11], they used the concept of triangle matrices for obtaining the minimal set of sufficient conditions of infinite series to be bounded. In 2017, Sonker and Munjal [12] found the approximation of the function $f \in Lip\ (\alpha, p)$ using infinite matrices of Cesàro submethod and in [13], they obtained boundness conditions of absolute summability factors. In this way by using the advanced summability method, we can improve the quality of the filters.

Borwein [3] extended many results on ordinary and absolute summability methods of integral. Çanak [4] and Totur [14] worked on the concept of Cesàro summability with a very interesting result for integrals. In the same direction, we extended the results of Mazhar [7] with the help of some new generalized conditions and absolute Nörlund summability $|N, p_n|_k$ factor for integrals.

II. KNOWN RESULTS

In [6], Kishore has proved the following theorem concerning |C, 1| and $|N, p_n|$ summability methods.

Theorem 1: *Let* $p_0 > 0$, $p_n \geq 0$ *and* p_n *be a non-increasing sequence. If* $\sum a_n$ *is summable* $|C, 1|$, *then the series* $\sum a_n P_n (n+1)^{-1}$ *is summable* $|N, p_n|$.

By concerning absolute Cesàro summability $|C, 1|_k$ factors and a positive monotonic non-decreasing function $\gamma(x)$, Özgen [8] obtained the following results for integrals.

Theorem 2: *Let* $\gamma(x)$ *be a positive monotonic non-decreasing function such that*

$$\lambda(x)\gamma(x) = O(1) \quad as \quad x \to \infty, \tag{10}$$

$$\int_0^x u \mid \lambda''(u) \mid \gamma(u) du = O(1), \tag{11}$$

$$\int_0^x \frac{\mid v(u) \mid^k}{u} du = O(\gamma(x)) \quad as \quad x \to \infty, \tag{12}$$

then the integrals $\int_0^\infty f(t) dt$ *is said to be summable* $|C, 1|_k$, $k \geq 1$.

III.MAIN RESULTS

In the present research article, we extended the result of Özgen [8] by using the $|C, 1|_k$ summability and some other concepts. With the help of functions and and Cesàro summability $|C, 1|_k$, we established the following theorem.

Theorem 3: *Let* $p(0) > 0$, $p(x) \geq 0$ *and* $p(x)$ *be a non-increasing function. Let* $\chi(x)$ *be a positive non-decreasing function and there be two functions* $\beta(x)$ *and* $\varepsilon(x)$ *such that*

$$\mid \varepsilon'(x) \mid \leq \beta(x), \tag{13}$$

$$\beta(x) \to 0 \quad as \quad x \to \infty, \tag{14}$$

$$\int_0^\infty u \mid \beta'(u) \mid \chi(u) du < \infty, \tag{15}$$

$$\mid \varepsilon(x) \mid \chi(x) = O(1), \tag{16}$$

$$\int_0^x \frac{\mid v(u) \mid^k}{u} du = O(\chi(x)) \quad as \quad x \to \infty, \tag{17}$$

then the integrals $\int_0^\infty f(t) dt$ *is said to be summable* $|N, p_n|_k$ *for* $k \geq 1$.

Note: The above theorem can be proved by using the concept of example that $\int_0^\infty x \mid \beta'(x) \mid \chi(x)dx < \infty$ is weaker $\int_0^\infty x \mid \varepsilon''(x) \mid \chi(x)dx < \infty$, and hence the introduction of the function $\{\beta(x)\}$ is justified.

Proof: It may be possible to choose the function $\beta(x)$ such that

$$|\varepsilon'(x)| \le \beta(x), \tag{18}$$

When $\varepsilon'(x)$ oscillates, $\beta(x)$ may be chosen such that $|\beta(x)| < |\varepsilon''(x)|$. Hence, $\beta'(x) << |\varepsilon''(x)|$, so that $\int_0^\infty x \mid \beta'(x) \mid \chi(x)dx < \infty$ is a weaker requirement than $\int_0^\infty x \mid \varepsilon''(x) \mid \chi(x)dx < \infty$.

IV. PROOF OF THE THEOREM

In order to prove the theorem, we need to consider only the special case in which $|N, p_n|_k$ is $|C, 1|_k$, that is, we shall prove that $\int_0^\infty f(t)dt$ is summable $|C, 1|_k$. Our theorem will then follow by means of theorem 1. Let $T(x)$ be the function of n^{th} $(C, 1)$ means of the integral $\int_0^\infty f(t)dt$. The integral is $|C, 1|_k$ summable, if

$$\int_0^x x^{k-1} \mid T'(x) \mid^k dx = O(1) \quad as \quad x \to \infty, \tag{19}$$

where $T(x)$ is given by

$$\begin{aligned} T(x) &= \frac{1}{x}\int_0^x\int_0^t \varepsilon(u)f(u)dudt \\ &= \frac{1}{x}\int_0^x \varepsilon(u)f(u)du\int_u^x dt \\ &= \frac{1}{x}\int_0^x (x-u)\varepsilon(u)f(u)du \\ &= \int_0^x \left(1-\frac{u}{x}\right)\varepsilon(u)f(u)du \end{aligned} \tag{20}$$

On differentiating both sides with respect to x, we get

$$\begin{aligned} T'(x) &= \frac{1}{x^2}\int_0^x u\varepsilon(u)f(u)du \\ &= \frac{\varepsilon(x)}{x^2}\int_0^x uf(u)du - \frac{1}{x^2}\int_0^x \varepsilon'(u)\int_0^u tf(t)dtdu \\ &= \frac{\varepsilon(x)v(x)}{x} - \frac{1}{x^2}\int_0^x u\varepsilon'(u)\left(\frac{1}{u}\int_0^u tf(t)dt\right)du \\ &= \frac{\varepsilon(x)v(x)}{x} - \frac{1}{x^2}\int_0^x u\varepsilon'(u)v(u)du \\ &= T_1(x)+T_2(x). \end{aligned} \tag{21}$$

Applying Minkowski's inequality,

$$|T_n|^k = |T_1 + T_2|^k < 2^k\left(|T_1|^k + |T_2|^k\right) \tag{22}$$

Applying Hölder's inequality, we have

$$\begin{aligned}
\int_0^x t^{k-1}\,|T_1(t)|^k\,dt &= \int_0^x t^{k-1}\frac{|v(t)|^k|\varepsilon(t)|^k}{|t|^k}dt \\
&= \int_0^x \frac{1}{t}|v(t)|^k|\varepsilon(t)|^{k-1}|\varepsilon(t)|\,dt \\
&\le \int_0^x \frac{|v(t)|^k}{t}|\varepsilon(t)|\,dt \\
&= |\varepsilon(x)|\int_0^x \frac{|v(t)|^k}{t}dt - \int_0^x |\varepsilon'(t)|\int_0^t \frac{|v(u)|^k}{u}dudt \\
&= \mathrm{O}(1)|\varepsilon(x)|\,\chi(x) - \int_0^x \beta(t)\chi(t)dt \\
&= \mathrm{O}(1) - \int_0^\infty |\beta'(x)|\,dx\int_0^x \chi(u)du \\
&\le \mathrm{O}(1) - \int_0^\infty u\,|\beta'(u)|\,\chi(u)du \\
&= \mathrm{O}(1)\ \ as\quad x\to\infty.
\end{aligned} \tag{23}$$

By virtue of the hypotheses of theorem 3,

$$\begin{aligned}
\int_0^x t^{k-1}\,|T_2(t)|^k\,dt &= \int_0^x t^{k-1}\frac{1}{t^{2k}}\left|\int_0^t u\varepsilon'(u)v(u)du\right|^k dt \\
&\le \int_0^x \frac{1}{t^2}\left(\int_0^t u^k\,|\varepsilon'(u)|^k|v(u)|^k\,du\right)\left(\frac{1}{t}\int_0^t du\right)^{k-1}dt \\
&= \int_0^x |u\varepsilon'(u)|^{k-1}|u\varepsilon'(u)||v(u)|^k\,du\int_u^x \frac{dt}{t^2} \\
&= \int_0^x |u\varepsilon'(u)||v(u)|^k\left(\frac{1}{u}-\frac{1}{x}\right)du \\
&\le \int_0^x |u\varepsilon'(u)|\frac{|v(u)|^k}{u}du \\
&= x\,|\varepsilon'(x)|\int_0^x \frac{|v(u)|^k}{u}du - \int_0^x (u\,|\varepsilon'(u)|)'\int_0^u \frac{|v(t)|^k}{t}dtdu \\
&= x\,|\beta(x)|\,\chi(x) - \int_0^x |\beta(u)|\,\chi(u)du - \int_0^x u\,|\beta'(u)|\,\chi(u)du \\
&\le \int_x^\infty u\,|\beta'(u)|\,\chi(u)du - \int_0^x |\beta(u)|\,\chi(u)du - O(1) \\
&= \mathrm{O}(1)\ \ as\quad x\to\infty.
\end{aligned} \tag{24}$$

On collecting (20)-(24), we have

$$\int_0^x t^{k-1}\,|T'(t)|^k\,dt = O(1)\quad as\quad t\to\infty, \tag{25}$$

Hence proof of the theorem is complete.

V. Corollaries

Corollary 1: *Let* $p(0) > 0$, $p(x) \geq 0$ *and* $p(x)$ *be a non-increasing function. Let* $\chi(x)$ *be a positive non-decreasing function such that*

$$\varepsilon(x)\chi(x) = O(1) \quad as \quad x \to \infty, \tag{26}$$

$$\int_0^\infty u \mid \varepsilon''(u) \mid \chi(u) du = O(1), \tag{27}$$

$$\int_0^x \frac{\mid v(u) \mid^k}{u} du = O(\chi(x)) \quad as \quad x \to \infty, \tag{28}$$

then the integrals $\int_0^\infty f(t)dt$ *is said to be summable* $|N, p_n|_k$ *for* $k \geq 1$.

Corollary 2: *Let* $p(0) > 0$, $p(x) \geq 0$ *and* $p(x)$ *be a non-increasing function and* $\varepsilon(x)$ *be a convex function such that* $\int \frac{\varepsilon(x)}{x} dx$ *is convergent. If f is bounded on* $[R, \log n, 1]$ *with index k, then* $\int_0^\infty f(t)dt$, *is summable* $|N, p_n|_k$ *for* $k \geq 1$.

Corollary 3: *Let* $p(0) > 0$, $p(x) \geq 0$ *and* $p(x)$ *be a non-increasing function and* $\varepsilon(x)$ *be a convex function such that* $\int \frac{\varepsilon(x)}{x} dx$ *is convergent. If f is bounded on* $[R, \log n, 1]$, *then* $\int_0^\infty f(t)dt$, *is summable* $|N, p_n|$.

Note: The above corollaries can be derived by taking the following assumptions in the main result,

(i) For corollary 1, we take $| \varepsilon'(x) | = \beta(x)$.

(ii) For corollary 2, we take $\chi(x) = \log(x)$ and $\varepsilon(x)$ as a convex function.

(ii) For corollary 3, we take $\chi(x) = \log(x)$, $k = 1$ and $\varepsilon(x)$ as a convex function.

VI. Conclusion.

The main result of this research article is an attempt to formulate the problem of absolute summability factor of integrals which make a more modified filter. Through the investigation, we concluded that the improper integral is absolute Nörlund summable under the minimal sufficient conditions. Further, this study has a number of direct applications in rectification of signals in FIR filter (Finite impulse response filter) and IIR filter (Infinite impulse response filter). In a nut shell, the absolute summability methods are a motivation for the researchers, interested in studies of improper integrals.

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