Absolute Summability Factor \( |N, p_n|^k \) of Improper Integrals

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Abstract—In this paper, we defined the summability for integrals and established a theorem on absolute Nörlund summability \( |N, p_n| \) factors of improper integral under sufficient conditions. Some auxiliary results (well-known) have also been deduced from the main results under suitable conditions.

Keyword - Absolute Summability, Nörlund summability, Improper Integrals, Inequalities for Integrals.

I. INTRODUCTION

1. Summability factor concerning infinite series: Let \( \sum_{n=0}^{\infty} a_n \) be an infinite series with sequence of partial sums, \( s_n = \sum_{n=0}^{n} a_n \) and \( \sigma_n \) be the \( n^{th} \) Cesàro means of the series, i.e.,

\[
\sigma_n = \frac{1}{n} \sum_{k=0}^{n} s_k
\]

(1)

The series \( \sum_{n=0}^{\infty} a_n \) is said to be \(|C, 1|, k \geq 1 \) summable \([5]\), if

\[
\lim_{n \to \infty} \sigma_n = s,
\]

(2)

and

\[
\sum_{n=1}^{\infty} p^{k-1} |\sigma_n - \sigma_{n-1}| < \infty
\]

(3)

2. Summability factor concerning improper integrals: Let \( f \) be a real valued continuous function of \( t \) in the interval \([0, \infty)\) and \( s(x) = \int_{0}^{x} f(t)dt \). Then, \( \int_{0}^{\infty} f(t)dt \) is said to be summable \(|C, 1|\), if

\[
\int_{0}^{\infty} |\sigma'(x)| \ dx < \infty.
\]

(4)

and \( \int_{0}^{\infty} f(t)dt \) is said to be summable \(|C, 1|, k \geq 1 \), if

\[
\int_{0}^{\infty} x^{k-1} |\sigma'(x)|^k \ dx < \infty,
\]

(5)

where \( \sigma(x) \) is Cesàro mean of \( s(x) \) and given by

\[
\sigma(x) = \frac{1}{x} \int_{0}^{x} (x - t) f(t)dt,
\]

(6)

or

\[
\sigma(x) = \frac{1}{x} \int_{0}^{x} s(t)dt.
\]

(7)

The Kronecker identity: \( s(x) - \sigma(x) = \nu(x) \), where

\[
\nu(x) = \frac{1}{x} \int_{0}^{x} tf(t)dt.
\]

(8)
The condition (5) can be written as
\[
\int_0^\infty \frac{1}{x} |v(x)|^k \, dx < \infty. \quad (9)
\]

Considering the \((N, P_n)\) and \((K, 1, \alpha)\) summability, Parashar [9] obtained the minimum set of conditions for an infinite series to be \((K, 1, \alpha)\) summable. In 1986, Bor [1] found the relationship between two summability techniques \(|C, 1|_k\) and \(|N, p|_{\alpha k}\) and in [2], he used the \(|N, p|_{\alpha k}\) for generalization of a theorem based on minimal set of sufficient conditions for infinite series. In 2016, Sonker and Munjal [10] determined a theorem on generalized absolute Cesàro summability with the sufficient conditions for infinite series and in [11], they used the concept of triangle matrices for obtaining the minimal set of sufficient conditions of infinite series to be bounded. In 2017, Sonker and Munjal [12] found the approximation of the function \(f \in Lip (\alpha, p)\) using infinite matrices of Cesàro submethod and in [13], they obtained boundness conditions of absolute summability factors. In this way by using the advanced summability method, we can improve the quality of the filters.


II. KNOWN RESULTS

In [6], Kishore has proved the following theorem concerning \(|C, 1|\) and \(|N, p|_{\alpha k}\) summability methods.

**Theorem 1:** Let \(p_0 > 0, p_n \geq 0\) and \(p_n\) be a non-increasing sequence. If \(\sum a_n \) is summable \(|C, 1|\), then the series \(\sum a_n P_n (n + 1)^{-1}\) is summable \(|N, p|_{\alpha k}\).

By concerning absolute Cesàro summability \(|C, 1|_k\) factors and a positive monotonic non-decreasing function \(\lambda(x)\), Özgen [8] obtained the following results for integrals.

**Theorem 2:** Let \(\lambda(x)\) be a positive monotonic non-decreasing function such that
\[
\lambda(x) = O(1) \quad \text{as} \quad x \to \infty, \quad (10)
\]
\[
\int_0^\infty u |\lambda''(u)| \gamma(u) \, du = O(1), \quad (11)
\]
\[
\int_0^\infty \frac{|v(u)|^k}{u} \, du = O(\gamma(x)) \quad \text{as} \quad x \to \infty, \quad (12)
\]
then the integrals \(\int_0^\infty f(t) \, dt\) is said to be summable \(|C, 1|_k, k \geq 1\).

III. MAIN RESULTS

In the present research article, we extended the result of Özgen [8] by using the \(|C, 1|_k\) summability and some other concepts. With the help of functions \(\varepsilon(x)\) and \(\chi(x)\) Cesàro summability \(|C, 1|_k\), we established the following theorem.

**Theorem 3:** Let \(p(0) > 0, p(x) \geq 0\) and \(p(x)\) be a non-increasing function. Let \(\chi(x)\) be a positive non-decreasing function and there be two functions \(\beta(x)\) and \(\varepsilon(x)\) such that
\[
|\varepsilon'(x)| \leq \beta(x), \quad (13)
\]
\[
\beta(x) \to 0 \quad \text{as} \quad x \to \infty, \quad (14)
\]
\[
\int_0^\infty u |\beta'(u)| \chi(u) \, du < \infty, \quad (15)
\]
\[
|\varepsilon(x)| |\chi(x)| = O(1), \quad (16)
\]
\[
\int_0^\infty \frac{|v(u)|^k}{u} \, du = O(\chi(x)) \quad \text{as} \quad x \to \infty, \quad (17)
\]
then the integrals \(\int_0^\infty f(t) \, dt\) is said to be summable \(|N, p|_{\alpha k}\) for \(k \geq 1\).
Note: The above theorem can be proved by using the concept of example that \( \int_0^\infty |x| \, dx < \infty \) is weaker than \( \int_0^\infty |x| \, dx < \infty \), and hence the introduction of the function \( \{ \beta(x) \} \) is justified.

Proof: It may be possible to choose the function \( \beta(x) \) such that
\[
|\varepsilon'(x)| \leq \beta(x),
\]
(18)
When \( \varepsilon'(x) \) oscillates, \( \beta(x) \) may be chosen such that \( |\beta(x)| < |\varepsilon''(x)| \). Hence, \( \beta'(x) \ll |\varepsilon''(x)| \), so that
\[
\int_0^\infty |\beta'(x)| \, dx < \infty
\]
is a weaker requirement than
\[
\int_0^\infty |\varepsilon''(x)| \, dx < \infty.
\]

IV. PROOF OF THE THEOREM
In order to prove the theorem, we need to consider only the special case in which \( |N, p_u| \) is \( |C, 1| \), that is, we shall prove that \( \int_0^\infty f(t) \, dt \) is summable \( |C, 1| \). Our theorem will then follow by means of theorem 1. Let \( T(x) \) be the function of \( n^{th} (C, 1) \) means of the integral \( \int_0^\infty f(t) \, dt \). The integral is \( |C, 1| \) summalbe, if
\[
\int_0^\infty |T'(x)|^k \, dx = O(1) \quad \text{as} \quad x \to \infty,
\]
(19)
where \( T(x) \) is given by
\[
T(x) = \frac{1}{x^2} \int_0^x f(u) \, du \, dt
\]
\[
= \frac{1}{x^2} \int_0^x f(u) \, du \int_0^x dt
\]
\[
= \frac{1}{x^2} \int_0^x (x-u) \, f(u) \, du
\]
\[
= \int_0^x \left( 1 - \frac{u}{x} \right) \, f(u) \, du.
\]
(20)
On differentiating both sides with respect to \( x \), we get
\[
T'(x) = \frac{1}{x^2} \int_0^x u \varepsilon(u) f(u) \, du
\]
\[
= \frac{\varepsilon(x)}{x^2} \int_0^x u f(u) \, du - \frac{1}{x^2} \int_0^x \varepsilon'(u) f(u) \, du
\]
\[
= \frac{\varepsilon(x) v(x)}{x} - \frac{1}{x^2} \int_0^x u \varepsilon'(u) \left( \frac{1}{u} f(t) \, dt \right) \, du
\]
\[
= \frac{\varepsilon(x) v(x)}{x} - \frac{1}{x^2} \int_0^x u \varepsilon'(u) \, du
\]
\[
= T_1(x) + T_2(x).
\]
(21)
Applying Minkowski’s inequality,
\[
|T_u|^k = |T_1 + T_2|^k \ll 2^k \left( |T_1|^k + |T_2|^k \right)
\]
(22)
Applying Hölder’s inequality, we have

\[
\int_0^x t^{k-1} |T_x(t)|^k dt = \int_0^x t^{k-1} \frac{|v(t)|^k |\varepsilon(t)|^k}{|t|^k} dt \\
\leq \int_0^x \frac{1}{t} |v(t)|^k |\varepsilon(t)|^k |t|^{-1} dt \\
\leq \int_0^x |v(t)|^k |\varepsilon(t)| dt \\
= |\varepsilon(x)| \int_0^x \frac{|v(t)|^k}{t} dt - \int_0^x \varepsilon'(t) \int_0^x \frac{|v(u)|^k}{u} du du dt \\
= O(1) |\varepsilon(x)| \chi(x) - \int_0^x \beta(t) \chi(t) dt \\
= O(1) - \int_0^x |\beta'(x)| du \chi(u) du \\
\leq O(1) - \int_0^x |u| \beta'(u) | \chi(u) du \\
= O(1) \text{ as } x \to \infty. \tag{23}
\]

By virtue of the hypotheses of theorem 3,

\[
\int_0^x t^{k-1} |T_x(t)|^k dt = \int_0^x t^{k-1} \frac{1}{t^k} \int_0^x u \varepsilon'(u) v(u) du du \right|^k dt \\
\leq \int_0^x \frac{1}{t^k} \left( \int_0^x u^{k-1} |\varepsilon'(u)|^k \right) \left( \int_0^x |v(u)|^k du \right)^k dt \\
\leq \int_0^x \frac{1}{u^k} \left( u^{k-1} |\varepsilon'(u)|^k \right) \left( |v(u)|^k \right) du \\
\leq \int_0^x u \varepsilon'(u) |v(u)|^k \left( \frac{1}{u} - \frac{1}{x} \right) du \\
\leq \int_0^x \frac{u \varepsilon'(u)}{u} \left( \frac{|v(u)|^k}{u} \right) du \\
= x \varepsilon'(x) \int_0^x \frac{|v(u)|^k}{u} du - \int_0^x (u \varepsilon'(u)) \frac{|v(u)|^k}{t} du du \\
= x |\beta(x)| \chi(x) - \int_0^x |\beta(u)| \chi(u) du - \int_0^x |u| \beta'(u) | \chi(u) du \\
\leq \int_0^x u \beta'(u) | \chi(u) du - \int_0^x |u| \beta'(u) | \chi(u) du - O(1) \\
= O(1) \text{ as } x \to \infty. \tag{24}
\]

On collecting (20)-(24), we have

\[
\int_0^x t^{k-1} |T_x(t)|^k dt = O(1) \text{ as } t \to \infty. \tag{25}
\]

Hence proof of the theorem is complete.
V. COROLLARIES

Corollary 1: Let \( p(0) > 0 \), \( p(x) \geq 0 \) and \( p(x) \) be a non-increasing function. Let \( \chi(x) \) be a positive non-decreasing function such that

\[
\varepsilon(x) \chi(x) = O(1) \quad \text{as} \quad x \to \infty,
\]

(26)

\[
\int_{0}^{\infty} u |\varepsilon'(u)| \chi(u) du = O(1),
\]

(27)

\[
\int_{0}^{\infty} \frac{v(u)^k}{u} du = O(\chi(x)) \quad \text{as} \quad x \to \infty,
\]

(28)

then the integrals \( \int_{0}^{\infty} f(t) dt \) is said to be summable \([N, p_n]\) for \( k \geq 1 \).

Corollary 2: Let \( p(0) > 0, p(x) \geq 0 \) and \( p(x) \) be a non-increasing function and \( \varepsilon(x) \) be a convex function such that \( \int \varepsilon(x) dx \) is convergent. If \( f \) is bounded on \([R, \log n, 1]\) with index \( k \), then \( \int_{0}^{\infty} f(t) dt \) is summable \([N, p_n]\) for \( k \geq 1 \).

Corollary 3: Let \( p(0) > 0, p(x) \geq 0 \) and \( p(x) \) be a non-increasing function and \( \varepsilon(x) \) be a convex function such that \( \int \varepsilon(x) dx \) is convergent. If \( f \) is bounded on \([R, \log n, 1]\), then \( \int_{0}^{\infty} f(t) dt \) is summable \([N, p_n]\).

Note: The above corollaries can be derived by taking the following assumptions in the main result,

(i) For corollary 1, we take \( |\varepsilon'(x)| = \beta(x) \).

(ii) For corollary 2, we take \( \chi(x) = \log(x) \) and \( \varepsilon(x) \) as a convex function.

(iii) For corollary 3, we take \( \chi(x) = \log(x) \), \( k = 1 \) and \( \varepsilon(x) \) as a convex function.

VI. CONCLUSION.

The main result of this research article is an attempt to formulate the problem of absolute summability factor of integrals which make a more modified filter. Through the investigation, we concluded that the improper integral is absolute Nörlund summable under the minimal sufficient conditions. Further, this study has a number of direct applications in rectification of signals in FIR filter (Finite impulse response filter) and IIR filter (Infinite impulse response filter). In a nut shell, the absolute summability methods are a motivation for the researchers, interested in studies of improper integrals.

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