More geometric tolerances on the same surface for a variational model

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***Abstract*— Tolerance analysis studies the accumulation of dimensional and/or geometric variations resulting from a stack-up of dimensions and tolerances. It is the fundamental tool to allocate the tolerances on the single components by solving the trade-off between the quality and the cost of the whole assembly.**

This work presents how it is possible to deal with more geometric tolerances applied to the same surface, when a tolerance analysis is carried out by means of a variational model. It modifies the expression of the stack-up function in order to deal with all geometrical tolerances, Enveloped and Independent principles. The proposed approach was applied to a case study constituted by three rigid parts. The obtained results show that all the applied geometrical tolerances significantly influences the assembly requirements and, therefore, they should not be ignored in the tolerance analysis of rigid assemblies.

Keyword-tolerance analysis, variational solid modelling, geometric tolerance, Envelope principle, Independent principle

I. INTRODUCTION

The actual global competition pushes industries to increase the quality of their products and, in the same time, to decrease the manufacturing costs. Mechanical products are usually made by assembling different parts, and their quality is guaranteed by the respect of some functional requirements assigned on the whole assembly. Being the dimensional and the geometrical tolerances applied to the assembly components, in order to limit the deviations from the nominal inevitably due to the manufacturing process, the respect of the functional requirements depends on the cumulative effect of the tolerances applied to the single components and on the assembly constraints.

The tolerance analysis has received considerable attention recently, and various methods have been proposed. Two main approaches can be distinguished [1]: one of them models the space allowed around nominal geometric entities, and another uses the parameterization of the deviation from theoretic geometric entities. Davidson and Shah [2] and Teissandier et al. [3] proposed the mathematical approaches of tolerances, which are essential to use the hypothetical Euclidean volume to represent all possible deviations in size, orientation and position of features. In reference a series of as-manufactured component models were generated within a NURBS-based solid modeling environment [4].

The methods discussed above used different mathematical formulations to simulate the actual surfaces of parts with geometrical deviations compared to the ideal surfaces, and then analyzed the effect of those manufacturing errors on the behaviors of the machines. They are suitable for the simple assemblies. For complicated ones, it is difficult to analyze the effects of the manufacturing errors on the final functional requirements.

As a result, a number of researchers developed some mathematical methods for tolerance analysis of rigid parts: vector loop, variational, matrix, jacobian, torsor [5].

Recently, the tolerance maps model was developed at the Arizona State University [6-7]. A Tolerance-Map (T-Map) is a hypothetical solid of points in n-dimensions which represent all possible variations of a feature or an assembly. Overlaying the coordinates of the T-Map the stack-up equations to perform the tolerance analysis are obtained. No one the models proposed by the literature is completely and univocally accepted and used.

A recently proposed concept for a paradigm shift in computer-aided tolerancing is the concept of Skin Model Shapes, which are specific variational part representatives employing discrete geometry schemes, such as point clouds and surfaces meshes. In a recent paper, approaches for the generation of such variational part representatives in discrete geometry have been presented, which are based on algorithms for the evaluation of geometric tolerances from point clouds and are capable of generating such representatives in conformance to pre-defined tolerance specifications [8].

This work focuses on variational model for tolerance analysis. A mathematical foundation of the variational model has been proposed by Martino and Gabriele [9], by Boyer and Stewart [10] and by Gupta and Turner [11].

Later, several additional variants have been proposed as well, and nowadays commercial CAT software packages are based on this approach, such as 3-DCS of Dimensional Control Systems® and VisVSA of UGS®. The basic idea of the variational model is to represent the variability of an assembly, due to tolerances and assembly constraints, through a parametric mathematical model. It models the dimensional and geometrical variations affecting a part by means of differential homogeneous transformation matrices. Marziale and Polini analyzed its advantages and limits in detail [12].

Variational model can handle almost all geometrical tolerances, form errors too [13]. It is suitable for both simple and complicated assemblies. However, for precision assemblies, the effect of more geometric tolerances applied to the same surface cannot be ignored. This paper presents how to model more geometric tolerances applied to the same surface of the assembly components in a Variational model to carry out tolerance analysis for a mechanical assembly. The proposed method translates each geometric tolerances into translational and rotational variations that are combined with those due to dimensional tolerances applied to the same surface. The obtained results on a case study have been compared with those due to two methods of the literature [14-15].

The rest of this article is organized as follows. The next section recalls the main features of the variational model. Then, the parameters of the models are modified to take into account more geometric tolerances applied to the same surface. The possibility to model both the Envelope and the Independent model is taken into account. An example is provided and its variational model is obtained and solved in great detail. The final section summarizes and concludes the study.

II. Variational Model for Tolerance Analysis

The aim of the tolerance analysis of an assembly is to evaluate the cumulative effect due to the tolerances, that are assigned to the assembly components, on the functional requirements of the whole assembly. Each functional requirement is schematized through an equation, that is usually called stack-up function, whose variables are the model parameters that are function of the dimensions and the tolerances assigned to the assembly components. It looks like

$$FR = f(p_1, p_2, \ldots, p_n) \tag{1}$$

where FR is the considered functional requirement, $p_1, \ldots, p_n$ are the model parameters and $f(p)$ is the stack-up function, that is usually not linear.

The basic idea of the variational model is to represent the variability of an assembly, due to the tolerances and the assembly conditions, through a set of parameters of a mathematical model.

To create an assembly, the designer has to define the nominal shape and the dimensions of each assembly component (these information are usually contained in CAD files). Then, the designer identifies the features of each component which affect the functional requirements (functional features) and the designer assigns the dimensional and geometrical tolerances to them. Each feature has its local Datum Reference Frame (DRF), while each component and the whole assembly have their own global DRF. In nominal condition, the homogeneous transformation matrix (called **TN**), that allows to pass from a DRF to another is known. When real features are machined, they depart from nominal. Assuming that real features maintain their nominal form (i.e. form deviations are neglected), the location of a real feature deviates from nominal, this deviation is expressed by parameters that constitute a differential homogeneous transformation matrix **DT**. To pass from the global DRF of the part i ($\mathbf{R}_i$) to the local DRF of a feature j of part i ($\mathbf{R}_{ij}$), it is enough to multiply the two matrices:

$$\mathbf{T}_{Ri\text{->}Rij} = \mathbf{TN}_{Ri\text{->}Rij} \cdot \mathbf{DT}_{Rij} \tag{2}$$

where $\mathbf{T}_{Ri\text{->}Rij}$ is the total transformation matrix to pass from the global DRF of the part i to the local DRF of feature j of part i; $\mathbf{TN}_{Ri\text{->}Rij}$ is the nominal transformation matrix to pass from the global DRF of the part i to the local DRF of feature j of part i; $\mathbf{DT}_{Rij}$ is the differential transformation matrix of the feature j of part i.

If a feature may not be directly referred to the global DRF, it is reported to it through a chain of features. To calculate the total matrix, it is enough to make the product of the single contributions as shown in Fig. 1 that is valid for the case of two transformations.

Once modeled the variability of the components, they have to be assembled together. The relative location of the parts is expressed by means of parameters (that are called *small kinematic adjustments*), which constitute the differential homogeneous transformation matrix **DA** (the transformation matrix is indicated by the letter *A*=assembly to distinguish it from the matrix **DT** that is for the part).

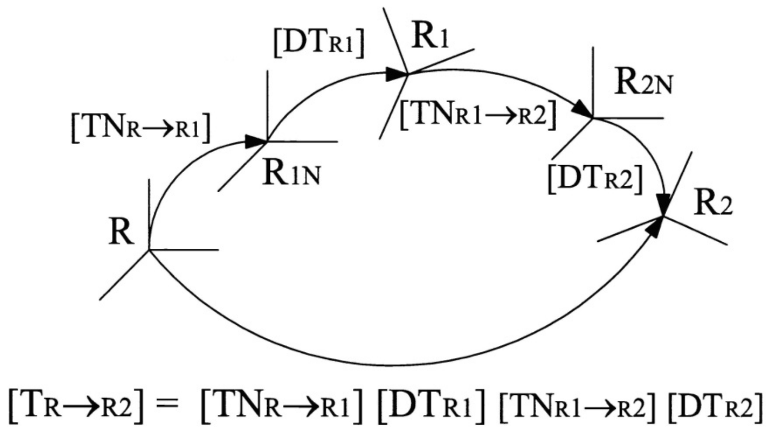

Fig. 1. Model of a stack-up function in a part

The total transformation to pass from the global DRF of part i ($\mathbf{R}_i$) to the global DRF of part l ($\mathbf{R}_l$), is simply obtained by means of the following equation (see Fig. 2):

$$\mathbf{A}_{Ri\text{->}Rl} = \mathbf{AN}_{Ri\text{->}Rl} \cdot \mathbf{DA}_{Ri\text{->}Rl} = \mathbf{TN}_{Ri\text{->}Rij} \cdot \mathbf{DT}_{Rij} \cdot \mathbf{DA}_{Rij\text{->}Rlk} \cdot \mathbf{DT}^{-1}_{Rlk} \cdot \mathbf{TN}^{-1}_{Rl\text{->}Rlk} \quad (3)$$

where: $\mathbf{A}_{Ri\text{->}Rl}$ is the assembly matrix between part i and part l, $\mathbf{AN}_{Ri\text{->}Rl}$ is the assembly matrix between part i and part l in nominal condition, $\mathbf{DA}_{Ri\text{->}Rl}$ is the differential assembly matrix between part i and part l, $\mathbf{DA}_{Rij\text{->}Rlk}$ is the differential assembly matrix between the feature j of part i and the feature k of part l, $\mathbf{TN}_{\mathbf{Ri\text{->}Rij}}$ is the nominal transformation matrix to pass from the global DRF of part i to the local DRF of feature j of part i, $\mathbf{DT}_{\mathbf{Rij}}$ is the differential transformation matrix of feature j of part i, $\mathbf{DT}_{\mathbf{Rlk}}$ is the differential transformation matrix of feature k of part l, and $\mathbf{TN}_{\mathbf{Rl\text{->}Rlk}}$ is the nominal transformation matrix to pass from the global DRF of the part l to the local DRF of feature k of part l. The differential assembly matrix $\mathbf{DA}_{\mathbf{Ri\text{->}Rl}}$ and $\mathbf{DA}_{\mathbf{Rij\text{->}Rlk}}$ are hard to evaluate, since they depend by both the tolerances, that are applied to the components in contact, and the assembly conditions.

Some are the works in the literature to evaluate these differential matrices. A strategy is to model the join between the coupled parts by reconstructing the coupling sequence between the features [16]. Another possibility is to impose some analytical constraints on the assembly parameters [17]. When all the transformation matrices are obtained, it is possible to express all the features in the same global DRF of the assembly (**R**); then the functional requirements on the assembly can be modelled. They appear as (1).

Once modelled the stack-up function, two are the approaches to solve it: a worst case or a statistical one [18]. To carry out a worst case analysis it is needed to define the worst configurations of the assembly that satisfy the variations due to the imposed tolerances. Therefore, it is needed to solve a problem of optimization (maximization and/or minimization) under constraints, in which the tolerances determine the structure of the ties [19]. To solve this problem the methods developed by the literature may be applied [20]. To carry out a statistical analysis it is needed to specify the different contributions to the deviation of a feature to which a tolerance is applied. This means that it is needed to specify the range of variation of the position, of the orientation and of the dimension of a feature due to the applied tolerances [21]. It often happens that those three contributions are not all known.

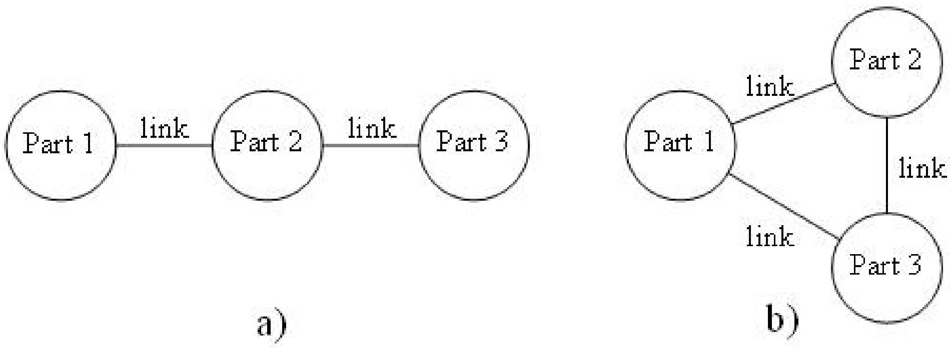

Fig. 2. a)Linear stack-up function and b)network stack-up function

Therefore, the simplifications usually adopted is to consider independent the three contributors and to assign to each of them a probability density function, usually Gaussian, able to cover a part, arbitrarily chosen, of the whole range of variation of the feature to which the tolerance is applied [15]. Therefore, a histogram of the functional requirement is obtained using a Monte Carlo simulation technique [22-23], and the evaluation of its variation range is assumed as ±3σ of the histogram (three sigma paradigm [17]).

III. Modelling More Tolerances Applied to the same Surface using a Variational Model

To assign a range to the parameters of a surface to which a set of tolerances has been allocated, the model should take into account the interaction of the tolerance zones.

The proposed model uses a diagram called *Allocation Flow Chart* (AFC). An AFC decomposes all of the possible deviations of the features into different contributors (Fig. 3): t_d is the range of the dimensional tolerance between two features (that are typically planes); t_f, t_p and t_o are the ranges of form, position and orientation tolerances respectively applied to the same features. The position tolerance zone t_p is considered applied to one of the two features, since the other is considered as reference (datum). The form t_f and the orientation t_o tolerance zones are decomposed into two contributors respectively for the two features, and for each feature along the x and y-axes. To each model parameter p_i a Probability Density Function (PDF) is assigned, that is assumed Gaussian, with standard deviation equal to one sixth of the corresponding tolerance range (t_{pi}) that may be extracted from *AFC*. The model parameters are considered independent.

When one or more tolerances are applied on the features, some of the tolerance zones showed in the *AFC* are known. The unknown tolerances may be determined by considering that the total effect of one or more tolerance zones applied on the same feature may be statistically accumulated as:

$$t_{tot}^2 = t_1^2 + t_2^2 + t_3^2 \qquad (4)$$

where t_{tot} is the total tolerance zone, and t_1, t_2, t_3, are the single tolerance zones. This model is based on the assumption that the parameters are dependent and distributed according to a Gaussian probability density function with average equal to the nominal value and standard deviation equal to one sixth of the applied tolerance range. If the total effect of the tolerances is known, the tolerance zone of each single parameter, may be calculated by considering that they have the same influence and, therefore:

$$t_1 = t_2 = t_3 = \frac{t_{tot}}{\sqrt{3}} \qquad (5)$$

However, if the total effect of the tolerances is known together with the range of variation of two contributors (say t_1 and t_2), the unknown contributor may be calculated by means of the following expression:

$$t_3 = \sqrt{t_{tot}^2 - t_1^2 - t_2^2} \qquad (6)$$

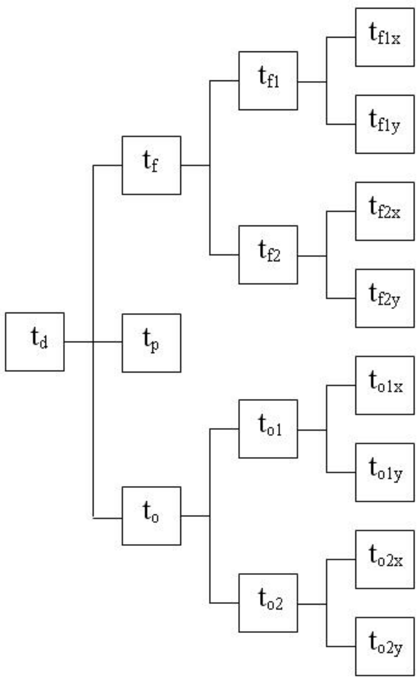

Fig. 3. *Allocation Flow Chart* in 2D case

The proposed model may be used to pass from the ranges of variation of the applied tolerances to the ranges of variation of the model parameters. For example Fig. 4 shows a dimensional tolerance and two geometrical tolerances (a planarity and a parallelism) assigned to the same two features (planes). The corresponding *Allocation Flow Chart* is shown in Fig. 5. It is obtained by proceeding from the known parameters to the unknown ones.

When the tolerance range of each model parameter has been evaluated, the proposed model assigns them a Gaussian probability density function with average equal to zero and standard deviation equal to one sixth of the tolerance zone thickness admitted by the *Allocation Flow Chart*. Therefore:

$$r_{x,y1,2} = r_{Dx,y1,2} + N\left(0, \frac{t_{o1,2x,y}}{3L_{y,x}}\right) \tag{7}$$

$$t_{z2} = N\left(0, \frac{t_p}{6}\right) \tag{8}$$

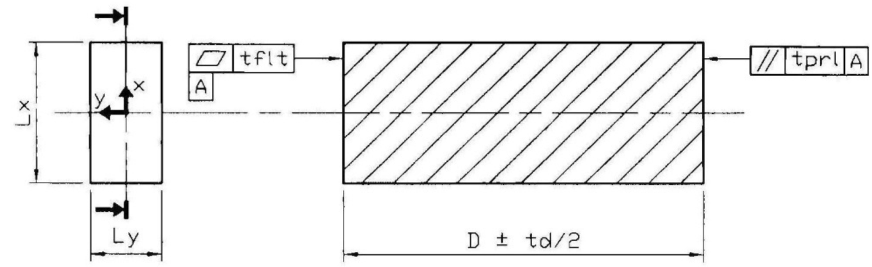

Fig. 4. Case study to apply the *Allocation Flow Chart*

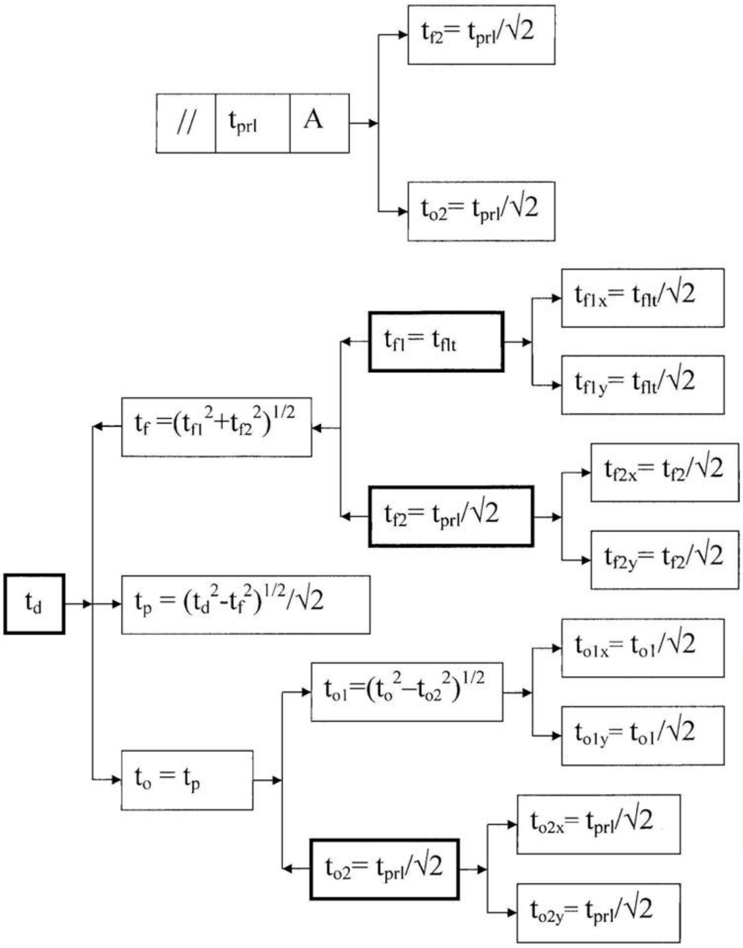

Fig. 5. *AFC* of the case study (Fig. 4)

$$p_{f1,2} = N\left(0, \frac{t_{f1,2}}{6}\right) \tag{9}$$

$$p_{f1,2x,y} = N\left(0, \frac{t_{f1,2x,y}}{6}\right) \tag{10}$$

where r_{x1}, r_{y1}, r_{x2}, r_{y2}, t_{z2}, p_{f1}, p_{f2} are the seven model parameters (the last two can be subdivided in p_{f1x}, p_{f1y}, p_{f2x}, p_{f2y} if necessary), t_{o1x}, t_{o1y}, t_{o2x}, t_{o2y}, t_p, t_{f1}, t_{f2} are the seven tolerance contributor obtained from the Allocation Flow Chart, and r_{Dx1}, r_{Dy1}, r_{Dx2}, r_{Dy2} are the rotation parameters of the datum. The rotation parameters have to be referred to a datum. If the datum is not clearly shown, the datum rotation have to be assumed as zero. For example in Figure 4 the datum of the second surface is constituted by the first one, the rotation parameters have to be set as $r_{Dx2}=r_{x1}$ and $r_{Dy2}=r_{y1}$.

IV. Modelling Envelope and Independent Principles using a Variational Model

To show how modelling Enveloped and Independence principles with a variational model, the case study of Fig. 6 has been considered. The dimensional tolerance is considered symmetric as regards the nominal value *D*.

The proposed approach removes the assumption of the literature model to consider one of the plane defining the dimensional tolerance perpendicular to the evaluation direction. Moreover, it is removed the assumption that the planes have no form deviations. Therefore, to characterize the distance between two planes, it is enough to take into account the vertices of the planes, as shown in the scheme of Figure 7 for the considered case study.

The proposed approach considers five parameters to represent the planes delimiting a 2D considered dimensional tolerance (see Fig. 7): the rotations r_{x1} and r_{y1} around the *x* and *y* axes of the plane *1*, the rotations r_{x2} and r_{y2} around the *x* and *y* axes of the plane *2*, the translation t_{z2} of the plane *2* along the nominal direction. It is assumed that $tang(r) \cong r$, since the rotations are usually small.

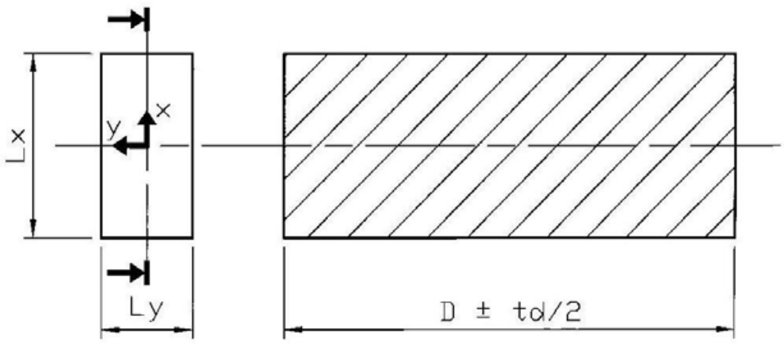

Fig. 6. A case study with dimensional tolerance

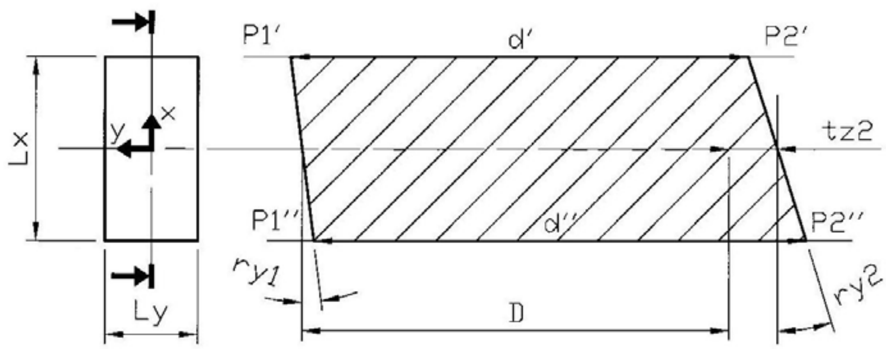

Fig. 7. The 2D proposed dimensional model

The translation effect between the planes is completely assigned to the second plane, since it is a simple way to represent the relative shift between the planes. The proposed model allows to model more than one tolerance applied to the same surface.

A further doubt remains on the interpretation of the concept of distance between the planes. In fact, the distance between the two planes depends by the considered evaluation direction, for example P1'P2' or P1'P2'' shown in Fig. 8. Therefore, by considering a dimensional tolerance according to the standards, the dimension between two planes may range between a minimum and a maximum value (see Fig. 9):

$d = [min(d_1,d_2,d_3,d_4), max\ (d_1,d_2,d_3,d_4]$ (11)

where d_1, d_2, d_3 and d_4 is the dimension evaluated along different directions.

By considering the deviations of the two boundary planes the dimensional tolerance implies that:

$D\text{-}t_d/2 \leq d = [min(d_1,d_2,d_3,d_4), max\ (d_1,d_2,d_3,d_4] \leq D+t_d/2$ (12)

where D is the nominal dimensional and t_d is the dimensional tolerance range.

Considering that usually the angles between the nominal and the effective measurement directions are small, it is possible to adopt the following simplifications: $d_1 = d_3 = d'$ and $d_2 = d_4 = d''$. Equation (12) becomes:

$D\text{-}t_d/2 \leq d', d'' \leq D+t_d/2$ (13)

It is important to note that the dimensional tolerance assigned to the distance between two planes does not limit the deviations of the orientation of the two planes. To limit the deviation of the features' orientation there are two possibilities: to add a further geometrical tolerance on the features or to apply the Envelope Principle to the dimensional tolerance. The application of the Envelope Principle means that the whole block has to be inside the block of perfect form at Maximum Material Condition (MMC). The application of the Envelope Principle implies an additional constraint that must be considered where a dimensional tolerance is modelled into a stack-up function of an assembly. Therefore, if a dimensional tolerance is applied without the Envelope Principle, this difference must be considered too. The proposed model allows to consider this difference, as described in the following paragraph.

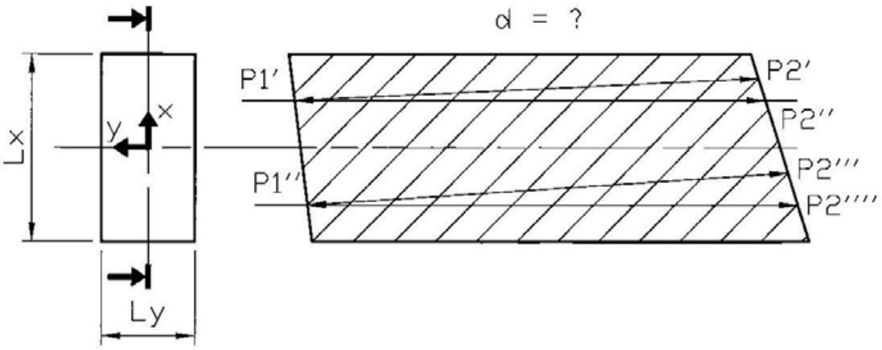

Fig. 8. Distance between the substitute planes

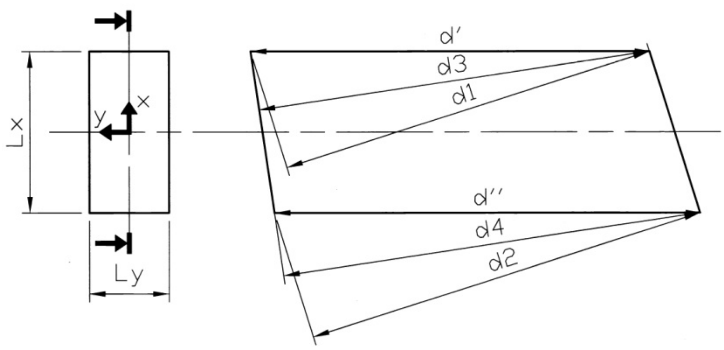

Fig. 9. Approximated distance between the substitute planes

A. Envelope Principle

To solve a stack-up function by a worst case approach, each tolerance involves a constraint that limits the possible variation of the features' parameters. For the 2D considered dimensional tolerance the proposed model implies that:

$$2 \cdot \left|t_{z_2}\right| + L_x \cdot \left|r_{y_2} - r_{y_1}\right| + L_y \cdot \left|r_{x_2} - r_{x_1}\right| \leq t_d \qquad (14)$$

where t_d is the range of dimensional tolerance, and this is the only constraint to impose when the Independence principle is applied.

If the Envelope principle is applied, the maximum material conditions should be satisfied; it implies the following additional constraint:

$$\pm 2 \cdot \left(t_{z_2}\right) + L_x \cdot \left(\left|r_{y_2}\right| + \left|r_{y_1}\right|\right) + L_y \cdot \left(\left|r_{x_2}\right| + \left|r_{x_1}\right|\right) \leq t_d \qquad (15)$$

where "+" is valid for pins (material inside the features) and "-" is valid for holes (material outside the features).

Therefore, the proposed model changes its constraints, if the Envelope or the Independence principles are applied.

V. CASE STUDY

The new developed approaches have been applied to the case study shown in Fig. 10. It is constituted by a box containing two disks. The aim of this work is the measurement of the gap g between the second disk and the top side of the box as a function of the tolerances applied to the components. Three cases have been considered. The first case takes into account only the dimensional tolerances; the Envelope Principle (according to ASME Y19.4 standard [24]) has been applied. The second case considers both the dimensional and the geometrical tolerances; the Envelope Principle has been applied. The last case considers both the dimensional and the geometrical tolerances and the Independent Principle (according to ISO 8015 standard [25]) has been applied. The case study has been solved through both the worst case and the statistical approaches.

The case study contains all the characteristics and the critical aspects of the problem, but in the same time it is so simple to calculate an exact geometric solution for the worst case analysis to compare the results of the models.

Table I shows the results obtained by means of the proposed variational model in comparison with the results due to other models of the literature on the same case study. The first two rows show the values of the gap range (Δg) due to the exact worst case approach [26]. The third, fourth and fifth rows show the results of the proposed Variational model. The last two rows show the results due to the Vector loop [27] and the Variational [12] models of the literature. The first column of results is due to the worst-case approach, while the second to a statistical approach that applies the root sum of squares or Monte Carlo simulations.

The worst case approach with only dimensional tolerances gives a below limit of the range (-0.78 mm) that is equal to that of the exact solution, since it is bounded by the Envelope principle, while the upper limit of the range (+1.03 mm) is smaller than the exact one. The same behaviour may be seen if both dimensional and geometric tolerances are considered together with the Envelope principle. If the Independence principle is applied, the new model gives an increase of the range according with the increase of the range of the exact solution.

A statistical approach has been implemented too in order to solve the stack-up functions of the tolerance analysis applied to this case study in order to obtain the tolerance of the g gap, once the parameters of each dimensional tolerance, represented through the new model, has been considered as independent statistical variables with Gaussian probability density functions. Once the stack-up functions of the assembly have been built and once the probability density functions have been assigned to each parameter of the model, a Monte Carlo simulation has been performed to solve the stack-up functions that has been implemented through Matlab® software package. The variation of the g gap has been estimated as $\pm 3\sigma$, where σ is the standard deviation of the g distribution obtained by Monte Carlo simulation.

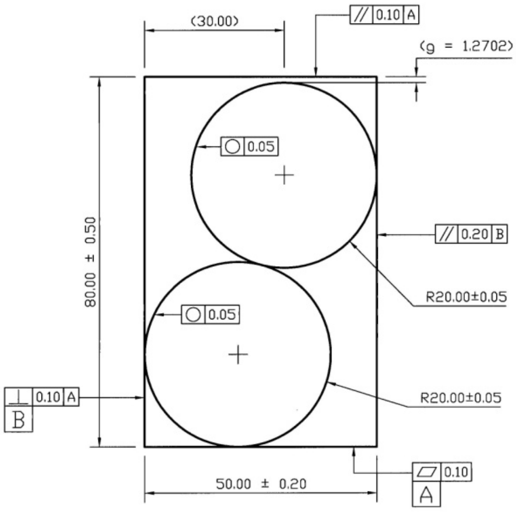

Fig. 10. 2D case study

If only the dimensional tolerances are applied, the new model implies a decrease of the range of the *g* gap. This effect can be justified considering that the proposed model uses many parameters, then the weight of the single one is less than in the actual model. Therefore, since the effects of form and orientation deviations increase, the effect of position deviations decrease (the total effect must be the same). The functional requirement is more influenced by the position deviations; therefore, its variability decreases.

When both dimensional and geometric tolerances are applied, the new model involves a decrease of the *g* gap. This effect is coherent with the consideration that if geometric tolerances are added to the parts, the variability must decrease.

The proposed model seems to give results nearer to the exact ones than the models of the literature with a worst-case approach. It gives results smaller than those of the literature models with a statistical approach.

VI CONCLUSIONS

This work shows how it is possible to model more geometric tolerances applied to the same surface into the tolerance analysis of mechanical rigid products by variational model.

The proposed approach uses a diagram called *Allocation Flow Chart* (AFC) to decompose all of the possible deviations of a feature into different contributors: that of the dimensional tolerance, that of form tolerance, that of position tolerance and that of orientation tolerance respectively applied to the same feature. The *Allocation Flow Chart* allows to pass from the tolerances applied to a feature to the model's parameters that are considered independent and they are statistically summed. To these model's parameters the statistical approach assign a Gaussian probability density function with average equal to zero and standard deviation equal to one sixth of the tolerance zone thickness admitted by the *Allocation Flow Chart.*

The developed approach schematizes a 2D dimensional tolerance applied to a feature by means of five parameters applied to the surfaces bounding the feature. In this way it is possible to consider more than one tolerances applied to the same surface. By using those five parameters, the single dimensional tolerance may be schematized through two different constraints, according to the application of the Envelope or the Independence principles, in the solution of the stack-up functions by means of the worst case approach.

The proposed approaches have been developed by means of Monte Carlo simulations in Matlab® environment. A case study is presented to evaluate the effectiveness of the proposed approach. In fact, the obtained results show that all the applied geometrical tolerances significantly influence the assembly functional requirement. This means that all the geometric tolerances should not be neglected when it deals with precision rigid mechanical assemblies. The proposed method allows to foresee real coupling among the parts, once all the geometrical tolerances have been allocated. Then, it is applied the Variational model to carry out the tolerance analysis of precision mechanical products. In this way, the design team may easily correct the issues, if the assembly functional requirements are not satisfied.

Table I

Results of Δg [mm]: comparison among different models.

Ref.	**Models**	**Δg [mm]**	
		Worst case	**Statistical**
[26]	Exact solution + Envelope Rule	+0,91 -0,78	-
[26]	Exact solution + Independence Rule	±0,91	-
NEW VARIATONAL	Dimensional tolerances & Envelope Principle	+1,03 -0,78	±0,43
	Geometric and dimensional tolerances & Envelope Principle	+0,80 -0,78	±0,39
	Geometric and dimensional tolerances & Independence Principle	+0,80 -0,79	±0,40
[27]	Vector loop	±1,03	±0,54
[12]	Variational	±0,78	±0,53

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