On the Resonance Conditions of Rigid Rocking Blocks

Claudia Casapulla [#1]
[#] Department of Structures for Engineering and Architecture, University of Napoli "Federico II", Italy
[1] casacla@unina.it

Abstract—The dynamic behavior of rigid free-standing blocks subjected to earthquake ground motions is highly non-linear and sensitive to small perturbations of various parameters. Many difficulties arise in defining reliable response spectra for such systems and these are well known in the literature.

This paper deals with the resonance conditions in order to highlight to what extent the ground motion details and the system parameters can influence the rocking response. The first step is the construction of an artificial input implying amplitude resonance for the motion, which is analyzed by means of a simplified equation of motion introduced by Housner (1963). The coefficient of restitution is assumed to be a variable of the problem to account also for other damping effects (e.g. local plastic deformations). A stabilized phase of the motion is identified for which an upper-bound of the maximum rotation angle of the block can be defined in closed form. The results are plotted in resonance spectra which point out the influence of the coefficient of restitution and the size and slenderness of the block. An interesting comparison with the response of an elastic damped SDOF oscillator in analogous resonance conditions is also presented.

Keyword- Rigid block dynamics, Multiple impulses, Rocking amplification, Size effect, Resonance spectra

I. INTRODUCTION

Historically, the rocking response of rigid blocks subjected to earthquake ground motion has been a field of interest to researchers for over a century. Nevertheless, it was Housner [1] who first gave the problem a modern treatment. He showed that despite the apparent simplicity of a single rocking block dynamics, a non-trivial behavior was present and a number of unexpected results emerged. Basically, the stability of a block subjected to a particular ground motion does not necessarily increase monotonically with the increasing size or decreasing slenderness ratio. Nor does the overturning of a block by a ground motion with particular intensity imply that the block will necessarily overturn under the action of more intense ground motion.

Thus, in order to simplify the analysis, Housner [1] described the base acceleration as a rectangular or a half-sine pulse and expressions were derived for the minimum acceleration required to overturn the block, as a function of the duration of the pulse. Experimental and numerical analyses were later developed by Yim et al. [2] showing that, in contrast with the response to a single pulse, the response to more irregular but simplified accelerograms is very sensitive to the geometrical parameters of the block, as well as to the details of ground motions and the coefficient of restitution. Therefore, they used a probabilistic approach to identify certain statistically recurrent properties of the response. Aslam et al. [3] also analyzed, both numerically and experimentally, the dynamic behavior of the block under harmonic excitations and simulated accelerograms, confirming the difficulty in providing prediction criteria for the response.

Priestley et al. [4] presented early experimental studies on a model slender structure and developed a practical methodology to compute displacements of the centre of gravity of the structure due to rocking motion by using standard displacement and acceleration response spectra. This was then adopted by the FEMA 356 document [5]. Makris and Konstantinidis [6] demonstrated that this methodology is oversimplified and does not take into account the fundamental differences in the dynamical structure of the two SDOF systems. They showed that the rocking spectrum is a distinct and valuable intensity measure of earthquakes and offers information on the earthquake shaking that is not identifiable by the response spectrum of an SDOF oscillator. As a consequence, rocking structures cannot be replaced by 'equivalent' SDOF oscillators.

However, from all of these works, see also [7-12], there emerges the fact that the problem of the stability against overturning of rigid blocks is still far from finding a general settlement and standard response spectra, generally used for elastic systems, are not suitable for such structures. Further, the specific source of rocking amplification remains largely unexplained. The main difficulties are still related to the description of the seismic input and the great sensitivity of the response to small variations in both system parameters and ground motion details. Moreover, when the rigid block model is taken as a basic reference for the seismic analysis of the out-of-plane mechanisms of masonry walls, further uncertainties related to specific aspects of the structural behavior of masonry need to be accounted for [13, 14].

This paper addresses the response of rocking structures to horizontal ground motion from the rigid body dynamics perspective. The primary goal is to explore the features of ground motions which would cause increasing amplification of rocking motion and to investigate the general trends of the rocking resonance. To achieve this, the attention is focused on the rocking response of a rigid free-standing rectangular block to an artificial sequence of instantaneous impulses, here called “resonance input”, which can continuously add energy to the system and cause rocking resonance.

In the following sections, the mathematical formulation for the rocking block is first presented, followed by an investigation of the optimized ground motion which can cause maximum energy input. The response to the single pulse before impacting the ground is analyzed by means of the simplified equation of natural motion first given by Housner [1] and suitable for relatively slender blocks. The expressions obtained in closed form, in terms of maximum rotation angle and duration of the half-cycle, are easily extended to the response to the pulse sequence. It will be shown that the response of the block to an unlimited resonance input tends towards a stabilized phase characterized by a periodic motion, provided that overturning is excluded. Numerical and parametric analyses are carried out with reference to the stabilized phase, together with the construction of resonance spectra showing the influence of the mean parameters (coefficient of restitution, slenderness and size of the block).

Similar optimization of the ground motion which can cause resonance conditions were also recently achieved by means of an energy approach by DeJong [15]. Multiple sinusoidal pulses with continuously decreasing frequency to represent resonance conditions were obtained numerically at each time step and relationships between the pulse period and the instantaneous rocking period was investigated. Despite some remarkable differences mainly due to the fact that the present analysis leads to expressions in closed form together with possible stabilized phases of response, the two analytical approaches reiterate that rocking structures do not have a fixed natural frequency and therefore cannot be forced at a single frequency which causes resonance.

Lastly, a useful comparison between the rocking response and the oscillatory response of an elastic damped single degree of freedom (SDOF) oscillator in resonance conditions is herein presented to highlight to what extent the amplitude resonance for the block is more intense with respect to that for the elastic damped SDOF oscillator.

II. THE MOTION OF THE RIGID ROCKING BLOCK AND THE RESONANCE INPUT

By neglecting the vertical components of the accelerations, the equation of rocking motion for a rigid free-standing rectangular block before the first impact, with positive signs of forces and angles indicated in Fig. 1, is obtained from D’Alembert’s Principle:

$$MgR\cos(\alpha+\theta)+I_O\ddot{\alpha}=M\ddot{y}R\sin(\alpha+\theta) \qquad (1)$$

where g is the acceleration of gravity, M is the mass, $I_O = 4\,M(a^2+b^2)/3$ is the corresponding moment of inertia (with respect to O), $R=\sqrt{a^2+b^2}$ is the half-diameter of the block, $\ddot{y}$ is the ground acceleration and α and $\ddot{\alpha}$ are the angular acceleration and displacement of the block, respectively.

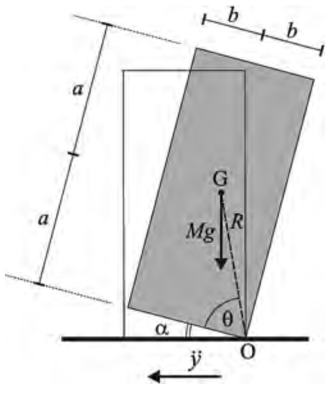


Fig. 1. Rigid rocking block system

The coefficient of friction is assumed to be sufficiently large as to prevent sliding between the block and the supporting base, while the dissipation of the energy due to impact of the block on the ground is represented by the coefficient of restitution. Within the classical rocking motion dynamics, the latter coefficient is commonly determined from the conservation of angular momentum and only depends upon the slenderness ratio, but independent of both the angular velocity before impact and the size of the block. This result could be easily recognized for the idealized conditions of rigid block and rigid base but, actually, the kinetic energy loss strictly depends on the materials of the block and the base, too. Therefore, in the following analysis the coefficient of restitution is assumed to be a variable of the problem to account also for other damping effects (e.g. local plastic deformations).

The optimization of the ground motion to maximize the rocking response can be represented by a particular sequence of instantaneous pulses which continuously add energy input to the system. The scheme in Fig. 2, proposed by Casapulla et al. [16], is adopted in this paper.

The first pulse is applied to the supporting base to start the motion. The subsequent pulses are applied right after each impact of the block on the ground ($\alpha = 0$), when the angular velocities are damped by the coefficient of restitution. The direction both of the pulses and the motion of the block are alternating, so as to input constant additional energy at each half-cycle. The interval time T_i between two subsequent pulses is progressively increasing as the durations of the half-cycles increase. The pulse sequence is assumed to be unlimited and the intensity of the pulses is $I = \ddot{y}\Delta t$, where $\ddot{y}$ is the ground acceleration and Δt is its duration.

This sequence applied to the block does imply a kind of amplitude resonance which can represent a threshold for its spectral response, as described in the following sections.

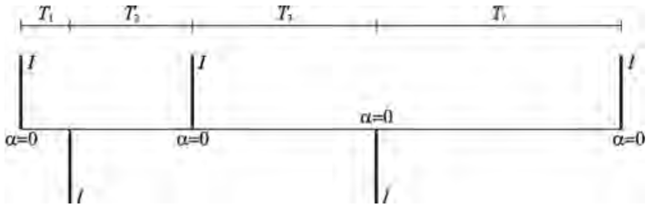


Fig. 2. Artificial seismic input as “resonance input” [16]

A. The Response to the Single Pulse

The simple case of a single pulse applied to the block is here analyzed and interesting observations can be derived regarding the most influencing parameters. The initial pulse *I* of the resonance input in Fig. 2 is assumed to be instantaneously applied at the base of the block (Fig. 1), so that it causes the velocity $\dot{\alpha}_1$ which is defined by the law of conservation of the angular momentum as:

$$\dot{\alpha}_1 = \frac{M\, I\, a}{I_O} = \frac{\lambda\, \ddot{\alpha}_g\, I}{g} \tag{2}$$

where $\lambda = a/b$ is the slenderness of the block and $\ddot{\alpha}_g$ is given by:

$$\ddot{\alpha}_g = \frac{3}{4}\frac{b}{a^2+b^2}\, g = \frac{3}{4b}\frac{1}{\lambda^2+1}\, g \tag{3}$$

From Eqs. (2) and (3) it is evident that the initial angular velocity is directly proportional to *I*, while it decreases for increasing values of block’s size *b* and slenderness λ.

For slender blocks with $\lambda \geq 3$, the natural motion of the block after the pulse can be derived by Eq. (1) and expressed by the well known approximated equation [1]:

$$\ddot{\alpha} + p^2(\alpha_c - \alpha) = 0 \tag{4}$$

in which $p = \sqrt{3g/(4R)}$ is the frequency parameter and $\alpha_c = \tan^{-1}(1/\lambda)$ is the maximum rotation angle of the block, whereas the half-diameter of the block can also be expressed as $R = \sqrt{a^2+b^2} = b\sqrt{1+\lambda^2}$. The larger and the slender the block is (larger *b* and λ), the smaller *p* is.

With the conditions $\alpha = 0$ and $\dot{\alpha} = \dot{\alpha}_1$ at time *t* = 0, the solution of Eq. (4), in terms of angular displacement and velocity, is given by the following expressions in closed form:

$$\begin{aligned} \alpha &= \alpha_c(1-\cosh pt) + \frac{\dot{\alpha}_1}{p}\sinh pt \\ \dot{\alpha} &= -\alpha_c\, p \sinh pt + \dot{\alpha}_1 \cosh pt \end{aligned} \tag{5}$$

which depend on parameters *b*, λ, and *I*, as well.

The maximum rotation angle is reached when $\dot{\alpha} = 0$, i.e.:

$$\alpha_{1\max} = \alpha_c\left[1-\cosh\left(\tanh^{-1}\frac{\dot{\alpha}_1}{\alpha_c\, p}\right)\right] + \frac{\dot{\alpha}_1}{p}\sinh\left(\tanh^{-1}\frac{\dot{\alpha}_1}{\alpha_c\, p}\right) \tag{6}$$

while the time at which the block completes an half-cycle and impacts the ground is:

$$\frac{T_1}{2} = \frac{2}{p} \tanh^{-1} \frac{\dot{\alpha}_1}{\alpha_c p} \tag{7}$$

In order to analyze the influence of the slenderness and the dimension of the block on its response to the single pulse of a given amplitude and duration, numerical results for Eqs. (6) and (7) are shown in Tables I and II, respectively. It is evident that the maximum rotation angle decreases for increasing values of the block's size and slightly increases with λ, while the duration of the half-cycle increases for increasing values of λ and is quite independent of b. The percentage difference between $T_1/2$ for a block with b = 0.2 m and that with b = 0.6 m increases with λ, although it remains relatively small (maximum 1.97%).

TABLE I
Values of α_{1max} (Eq. (6)) in function of λ and b (I = 0.15m/s)

	λ = 3	**λ = 5**	**λ = 8**	**λ = 10**
2*b* = 0.4 m	3.83e-3 rad	4.15e-3 rad	4.30e-3 rad	4.35e-3 rad
2*b* = 0.6 m	2.55e-3 rad	2.76e-3 rad	2.85e-3 rad	2.88e-3 rad
2*b* = 0.8 m	1.91e-3 rad	2.06e-3 rad	2.13e-3 rad	2.15e-3 rad
2*b* = 1.0 m	1.53e-3 rad	1.65e-3 rad	1.70e-3 rad	1.72e-3 rad
2*b* = 1.2 m	1.27e-3 rad	1.37e-3 rad	1.42e-3 rad	1.43e-3 rad

TABLE II
Values of $T_1/2$ (Eq. (7)) in function of λ and b (I = 0.15m/s)

	λ = 3	**λ = 5**	**λ = 8**	**λ = 10**
2*b* = 0.4 m	0.0909 sec	0.1541 sec	0.2498 sec	0.3145sec
2*b* = 0.6 m	0.0906 sec	0.1533 sec	0.2478 sec	0.3113 sec
2*b* = 0.8 m	0.0905 sec	0.1530 sec	0.2468 sec	0.3098 sec
2*b* = 1.0 m	0.0905 sec	0.1528 sec	0.2463 sec	0.3088 sec
2*b* = 1.2 m	0.0904 sec	0.1526 sec	0.2459 sec	0.3082 sec
Diff: $T_1/2(b_{min})$ vs. $T_1/2(b_{max})$	0.53%	0.94%	1.56%	1.97%

If the intensity of the initial pulse does not overturn the block, these results can easily be extended to the case of the pulse sequence and the same trends can be recognized, as described in the following section.

Basically, once a block starts rocking under a single pulse, it may or may not overturn depending on the magnitude of $\ddot{y}$ and the duration Δt. The intensity of the critical pulse required to overturn the block is generally given by the condition that the maximum rotation angle achieves the critical one, i.e. $\alpha_{1max} = \alpha_c = \tan^{-1}(1/\lambda)$. Incidentally, this is strictly true only when the block is under static loadings but not necessarily under dynamic conditions, especially when vertical accelerations are considered [2], [17]. The boundary between overturning and stable regions for a given geometry could easily be drawn, according to others [1], [2], [18]. Generally, larger accelerations or longer durations of the pulse are required to overturn larger blocks, and smaller accelerations or shorter durations are required to overturn relatively slender blocks.

However, what is most interesting to investigate herein is that a succession of smaller pulses can be more damaging than one larger pulse [1]. In fact, once a block starts rocking under an earthquake there is an energy build-up in the system as the block is subjected to successive pulses and seemingly small differences in the details of subsequent ground motion could greatly affect the response of the block. If the subsequent motion provides additional energy, although small, at the right time it could be sufficient to overturn the block, and this can also occur at much smaller peak accelerations than those predicted by a single pulse of given duration. Thus, the single pulse solution is of limited value when considering the rocking and overturning response of the block to arbitrary ground motions.

The “resonance input” herein proposed does follow this trend, since it is characterized by a sequence of instantaneous pulses which input constant additional energy at each half-cycle. This sequence can be recognized as an upper-bound of the possible dynamical loadings.

B. The Response to the “Resonance Input”

The results of the block response to the single pulse can easily be extended to the case of the pulse sequence. With the assumption of instantaneous pulse, the motion of the block between two impacts is of natural type and is characterized by a given initial velocity (the previous one damped by the coefficient of restitution during impact plus the new one due to the subsequent pulse).

Immediately after the first impact, the subsequent pulse of amplitude *I* (Fig. 2) acts on the block, increasing the initial velocity of the natural motion for the new half-cycle. The dissipation of energy due to impact on the ground is represented by the coefficient of restitution *C*, assumed as a variable of the problem. Thus, the absolute value of the angular velocity at the beginning of half-cycle *i* (*i* = 2, 3,.....*n*) is:

$$\dot{\alpha}_i = \dot{\alpha}_{i-1}C + \dot{\alpha}_1 = \dot{\alpha}_1\left[\left(1+\sum_{j=1}^{i-1}C^j\right) - C^{i-1}\right] \tag{8}$$

Generalizing Eq. (7), the duration of half-cycle *i* is:

$$\frac{T_i}{2} = \frac{2}{p}\tanh^{-1}\frac{\dot{\alpha}_i}{\alpha_c p} \tag{9}$$

It is easy to verify from Eqs. (8) and (9) that the duration of the generic half-cycle *i* is quite independent of the block dimension, as was already the case for a single pulse. In fact, keeping fixed *C* and *I*, its value is basically affected by the slenderness ratio and the number of the cycles which occurred before. Moreover, iterating Eq. (8) for $i \to \infty$, the angular velocity does not increase indefinitely but converges to the limit value $\dot{\alpha}^*$ given by:

$$\dot{\alpha}^* = \lim_{i\to\infty}\dot{\alpha}_i = \frac{\dot{\alpha}_1}{1-C} \tag{10}$$

This case implies that, when overturning does not occur before, also the duration of the half-cycles tends to stabilize to the value expressed by:

$$\frac{T^*}{2} = \frac{2}{p}\tanh^{-1}\frac{\dot{\alpha}^*}{\alpha_c p} \tag{11}$$

while the maximum rotation angle will be limited to the value:

$$\alpha^*_{\max} = \alpha_c\left[1-\cosh\left(\tanh^{-1}\frac{\dot{\alpha}^*}{\alpha_c p}\right)\right] + \frac{\dot{\alpha}^*}{p}\sinh\left(\tanh^{-1}\frac{\dot{\alpha}^*}{\alpha_c p}\right) \tag{12}$$

These results mean that the motion of the block becomes of periodic type, with period *T** which is now independent of the number of cycles because of constant angular velocity at each half-cycle, according to Eq. (10). Also, as observed for the response to the single pulse, it is still easy to verify that the size of the block strictly affects the stabilized angular velocity and displacement, but not the stabilized period, while the slenderness shows its influence on all motion parameters. As an example, in Fig. 3, which reports the stabilized periods vs. slenderness for *I* = 0.15 and *C* = 0.6, the curves for different block dimensions are very close each other with a small percentage difference increasing with λ, while the stabilized period sharply increases with λ. This trend confirms what already highlighted for the response to the single pulse in Table II.

Thus, the rocking response of the block to the proposed “resonance input” is influenced by both the system parameters and the ground motion properties, according to a systematic trend expressed by formulations in closed form. The different weights of each parameter will be examined in Section III.

Lastly, the amplification of the maximum rotation angle due to the pulse sequence with respect to the single pulse ground acceleration is derived from Eqs. (12) and (6). It can be demonstrated that this amplification, named $\zeta = \alpha^*_{\max}/\alpha_{1\max}$, strongly depends on the coefficient of restitution and, in particular, it increases sharply with the increasing of *C*. In fact, comparing the curves in Figs. 4a) and 4b), corresponding to block dimensions of 2*b* = 0.4 m and 2*b* = 1.2 m, respectively, it can be observed that the amplification is strictly affected by *C*, but it is quite independent of λ and slightly dependent on *b* for larger values of *C*.

In order to investigate the nature of such an amplification, in the last section this result is compared with the response of an equivalent elastic damped system to an analogous artificial input which implies a resonance condition.

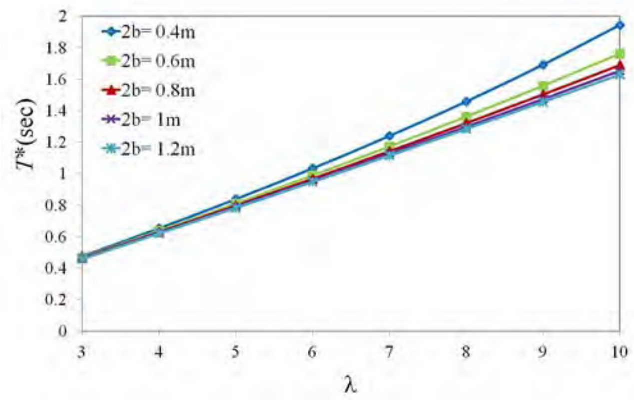
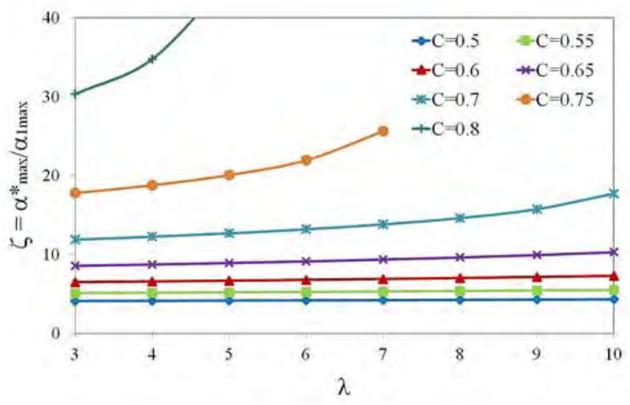
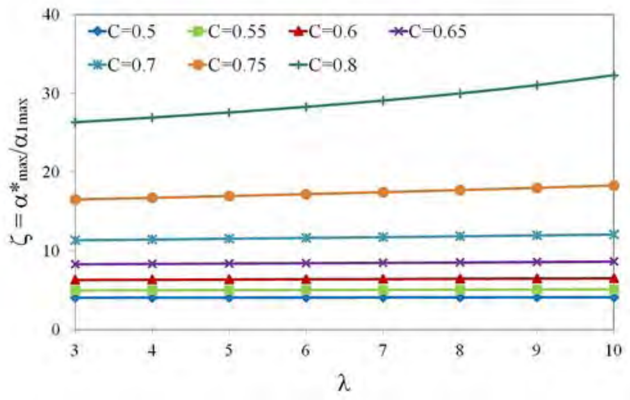


Fig. 3. Relation between the slenderness λ and the stabilized period T^*, for I = 0.15, C = 0.6 and different values of b



a)



b)

Fig. 4. Amplification ratio ζ vs. λ, for I = 0.15, different values of C and a) 2b = 0.4 m and b) 2b = 1.2 m

III. Parametric Analysis of the Results

In this section a parametric analysis of the response of a masonry rigid free-standing block to a "resonance" pulse sequence of unlimited duration is presented. The attention is focused on the stabilized phase of the response, when the duration of the cycles of the motion becomes constant and the block exhibits a periodic motion, with period T^*. The attainment of this phase is possible only for the blocks which features of slenderness, size and coefficient of restitution allow to exclude the overturning.

The reference to the stabilized phase allows to define a kind of resonance spectra for masonry rigid blocks in terms of the ratio α^*/α_c where α_c is the limit condition of the block assumed to be $\alpha_c = \tan^{-1}(1/\lambda)$. This kind of spectra is characterized by the fact that the period T^* of the stabilized cycles depends only upon the slenderness ratio and the coefficient of restitution *C*, and is independent of the size, as already observed within Eq. (11) and on Fig. 3.

All the results herein refer to a sequence of pulses with intensity *I* = 0.15 m/sec and to limited ranges of λ and 2*b*, i.e. $3 \le \lambda \le 10$ and $0.4 \text{ m} \le 2b \le 1.2$ m.

A. Characterization of the Stabilized Period

Fig. 5 shows the relation between the slenderness λ and the stabilized period *T**, for the given value of *I*, a medium size of block (2*b* = 0.8 m) and different values of *C*. Obviously, as this relation is representative of the stabilized period, it is meaningful only if the overturning does not occur and this depends on the size of the block for each given *C*, as will be shown later within the resonance spectra. So, from Fig. 5 it is firstly evident that, for a fixed *C*, *T** increases for increasing values of λ, suggesting that ground motions with large dominant periods are more threatening to slender blocks than to thick ones. This is an important aspect to be investigated, as the literature only indicates that long-period earthquakes may raise the risk of overturning a generic block [17].

In general, results indicate that energy amplification due to multiple distinct impulses can be important. Basically, earthquake ground motions do not tend to resemble such an artificial resonance input, so the possibility of true rocking resonance over a long period is unlikely. However, ground motions could contain a decreasing dominant frequency over a short time span, which would temporarily cause resonance effects on slender or thick blocks.

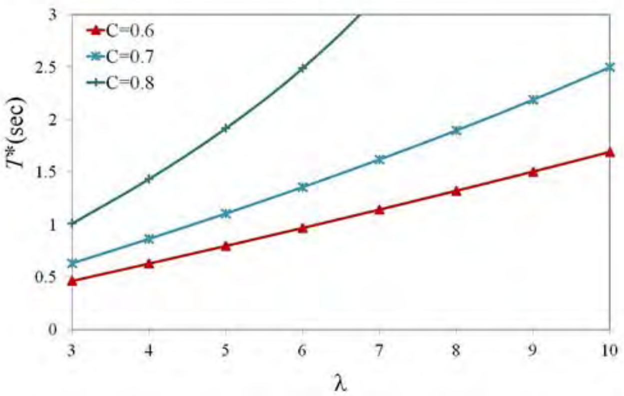


Fig. 5. Relation between the slenderness λ and the stabilized period *T**, for 2*b* = 0.8 m and different values of *C*

Moreover, *T** corresponding to a fixed value of λ increases as *C* increases. An increasing value of *C* implies that the damping effect on the velocity due to the impact on the ground decreases and therefore the initial velocity of the cycles within the stabilized phase and their duration increases, as evident from Eqs. (10) and (11). This means that the stabilized period increases as the damping effects decrease and this is the exact opposite of what happens for elastic systems if *T** is assumed as the fundamental damped period. This is another aspect to be investigated in further developments.

B. Influence of System Parameters

Apart from the features of the resonance pulse sequence herein taken as fixed, the meaningful parameters that influence the stabilized response are the coefficient of restitution, the size and the slenderness ratio of the blocks. These three parameters should reflect the various trends identified in the course of the analysis and in this section a kind of resonance spectra are proposed to summarize the results.

These are represented in Figs. 6, 7 and 8 in terms of α^*_{max}/α_c with reference to the same range of λ ($3 \le \lambda \le 10$) as used in Fig. 5, for fixed values of *C* and different sizes of the base (size effect).

The curves in Fig. 6 firstly show that for *C* = 0.6 all the considered blocks reach the stabilized phase without overturning. However, the ratio α^*_{max}/α_c, which describes the stabilized response, is strictly influenced by the size of the base and the slenderness ratio. Specifically, the ratio α^*_{max}/α_c displays a definite tendency to increase both for decreasing size of the base and almost proportionally with the slenderness ratio. These trends occur for any value of *C*, as shown in all the plotted results (Figs. 6, 7 and 8).

On the other hand, when *C* increases the size effect not only implies a higher ratio α^*_{max}/α_c for a given λ, but also has a determining role on the possibility of attainment of the stabilized phase. As an example, for *C* = 0.8 the blocks with base 2*b* = 0.6 m reach the stabilized phase only if $\lambda \le 7$, while overturning in the other cases.

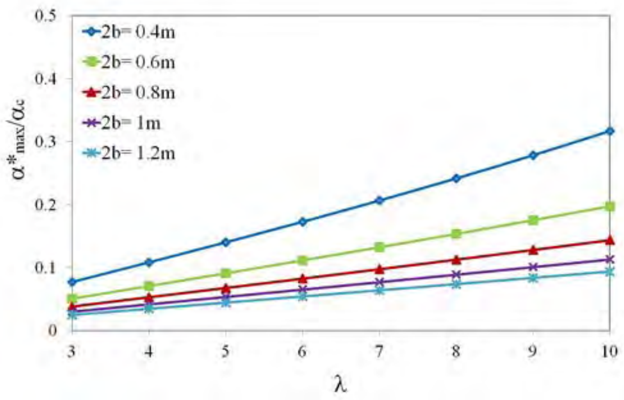


Fig. 6. Resonance spectra in terms of the ratio α^*_{max}/α_c, for $C = 0.6$ and different sizes of the base

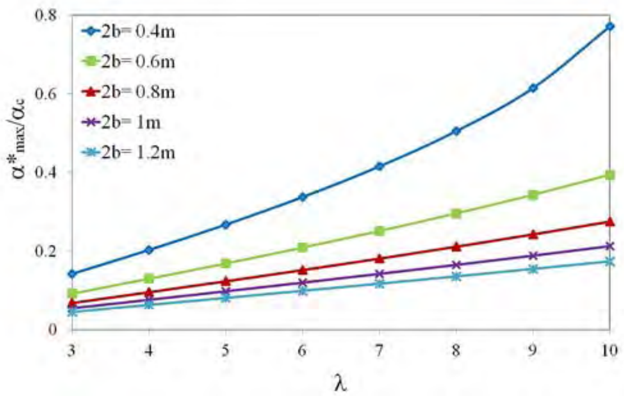


Fig. 7. Resonance spectra in terms of the ratio α^*_{max}/α_c, for $C = 0.7$ and different sizes of the base

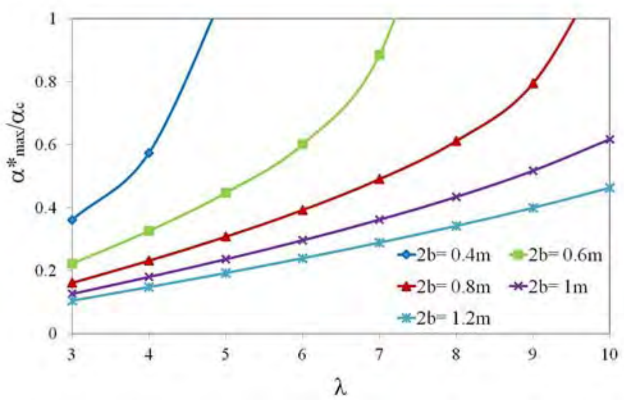


Fig. 8. Resonance spectra in terms of the ratio α^*_{max}/α_c, for $C = 0.8$ and different sizes of the base

For $C = 0.8$ the size effect on the stability is accentuated: in fact, only the blocks with $2b > 0.8$ m are safe from overturning for any value of the slenderness ratio up to 10. The smaller blocks ($2b < 0.8$ m) reach the stabilized phase only for values of slenderness decreasing with the decreasing of the base.

IV. A Comparison With The Elastic Damped System

The comparison between the oscillatory response of a single degree of freedom (SDOF) oscillator (regular pendulum) and the rocking response of a slender rigid block (inverted pendulum) was examined in depth by Makris and Konstantinidis [6]. They concluded that the elastic SDOF oscillator and the rocking block are two fundamentally different dynamical systems and the response of one should not be used to draw conclusion on the response of the other.

In particular, two orders of matter should be taken into account:

- the difficulty in defining the natural period of oscillation for the masonry rigid system;
- the difficulty in quantifying the damping level by means of the coefficient of restitution.

As for the first question, the proposed analysis of the response of the block to the sequence of pulses indicates that, in the presence of damping ($C < 1$), the possibility of recognizing a periodic motion is connected to the attainment of the stabilized phase, when the resonance effects are mitigated by the coefficient of restitution. In all other cases, the instantaneous rocking frequency is amplitude dependent.

As far as the second matter is concerned, Priestely et al. [4] proposed a simple formulation of viscous damping ratio for the rocking block equivalent to that for the elastic system by deriving it from the similarity between the two systems in the logarithmic decrement of the amplitude during a free vibration regime. Makris and Konstantinidis [6] later proposed the following empirical equation for the equivalent viscous damping ratio to approximate the formulation given in [4]:

$$\beta = -0.34\ln(C) \tag{13}$$

which is independent of the initial conditions and the number of cycles.

Indeed, interesting comments about this issue can be derived from the comparison between the oscillatory response and the rocking response to the “resonance input”, as discussed in the next section.

A. Response of the SDOF Oscillator to the Resonance Pulse Sequence

The formulations obtained by the application of the resonance pulse sequence to rigid blocks are now compared with the response of elastic systems to the same input. To this end, let us consider the damped linear elastic SDOF oscillator subjected to the pulse sequence presented and discussed above. Each pulse is still applied just after the system takes the configuration with null displacement.

The first pulse I of the artificial input in Fig. 2 instantaneously causes the velocity of the elastic system:

$$\dot{x}_1 = I \tag{14}$$

and the motion, which becomes of natural type, is governed by the classical differential equation:

$$\ddot{x} + 2\xi\omega\dot{x} + \omega^2 x = 0 \tag{15}$$

Starting from the initial configuration with null displacement and velocity given by Eq. (14), the solution of Eq. (15), in terms of displacement and velocity, is given by:

$$\begin{aligned} x &= e^{-\xi\omega t}\frac{\dot{x}_1}{\Omega}\sin\Omega t \quad \Omega = \omega\sqrt{1-\xi^2} \\ \dot{x} &= -\xi\omega e^{-\xi\omega t}\frac{\dot{x}_1}{\Omega}\sin\Omega t + e^{-\xi\omega t}\dot{x}_1\cos\Omega t \end{aligned} \tag{16}$$

Thus, at the time $t_1 = \dfrac{\pi}{\Omega}$, which identifies the half-cycle, it will be:

$$\begin{aligned} x_2 &= 0 \\ \dot{x}_2 &= -e^{-\xi\omega\frac{\pi}{\Omega}}\dot{x}_1 = -B\dot{x}_1 = -BI \end{aligned} \tag{17}$$

where:

$$B = e^{-\frac{\xi\pi}{\sqrt{1-\xi^2}}} \tag{18}$$

is the absolute ratio between the velocity at the end of the half-cycle and that at the beginning of the subsequent one; it has therefore the same meaning of the coefficient of restitution C, with the difference that for the elastic system the damping effect occurs along the whole duration of the cycle.

It is noteworthy that for lightly damped systems Eq. (18) furnishes the viscous damping ratio in function of B:

$$\xi = -\frac{1}{\pi}\ln(B) \tag{19}$$

after approximating $\sqrt{1-\xi^2}$ with one.

By comparing Eqs. (13) and (19) it results that the equivalent viscous damping ratio proposed by [6] is very close to the viscous damping ratio derived for the elastic system if it is assumed that $B = C$. Actually, although this would appear as a further validation of the empirical expression of the equivalent viscous damping ratio, the coefficient B and C cannot be used with the same practice, as discussed in the next section.

On the other hand, it is evident that, also for elastic systems, the proposed sequence of pulses, applied with alternative sign every time that $x = 0$ (every half-cycle), is a resonant sequence. Thus, by analogy with rigid block analysis, the absolute value of the displacement velocity at the beginning of the half-cycle i (i = 2, 3,.....n) is:

$$\dot{x}_i = \dot{x}_{i-1}B + \dot{x}_1 = \dot{x}_1\left[\left(1+\sum_{j=1}^{i-1}B^j\right) - B^{i-1}\right] \tag{20}$$

and its limit value will be:

$$\dot{x}^* = \lim_{i\to\infty}\dot{x}_i = \frac{\dot{x}_1}{1-B} \tag{21}$$

which is similar to Eq. (10).

As a consequence, the maximum displacement will not increase indefinitely but is limited to the value $x^*_{\max}$ corresponding to the velocity $\dot{x}^*$ and coming from the first of Eqs. (16).

Then, as the maximum displacement during the first half-cycle under a pulse of intensity I is:

$$x_{1\max} = e^{-\frac{\xi\omega\pi}{2\Omega}}\frac{\dot{x}_1}{\Omega} = e^{-\frac{\xi\omega\pi}{2\Omega}}\frac{I}{\Omega} \tag{22}$$

it is easy to verify that the ratio between the two displacements will be:

$$\psi = \frac{x^*_{\max}}{x_{1\max}} = \frac{1}{1-B} \tag{23}$$

B. Comparison between Rocking and Oscillatory Responses

The comparison of the amplification factor for elastic system expressed by Eq. (23) with that for rocking response ζ is indicated in Table III, considering a block with slenderness λ = 4 and block dimension 2b = 1.2 m. As already shown in Fig. 4, the amplification for rocking block is quite independent of the slenderness and block dimensions.

TABLE III
Comparison between the two amplification factors of response ψ (elastic oscillator) and ζ (rocking rigid block)

Elastic oscillator			**Rocking rigid block**		
B	**ξ (19)**	**ψ (23)**	***C***	**β (13)**	**ζ (Fig. 4)**
1	0	∞	1	0	Overturning
0.97	0.01	32.33	0.95	0.017	Overturning
0.91	0.03	11.11	0.9	0.036	182.37
0.85	0.05	6.87	0.85	0.055	51.60
0.80	0.07	5.05	0.8	0.076	26.91
0.75	0.09	4.05	0.75	0.098	16.72
0.71	0.11	3.41	0.7	0.121	11.44
0.66	0.13	2.96	0.65	0.146	8.33
0.62	0.15	2.64	0.6	0.174	6.34
0.58	0.17	2.39	0.55	0.203	4.99

The most interesting aspect emerging from the comparison in Table III is that, for commonly used values of the coefficient of restitution and of the viscous damping ratio, the amplitude resonance for the block is much more intense than that for the SDOF oscillator. As an example, considering the viscous damping ratio ξ = 5% commonly used for elastic systems, the amplification for the oscillator would be ψ = 6.87, while considering the equivalent damping ratio for rocking rigid block β = 5% corresponding to the coefficient of restitution $C \approx 0.85$, the amplification would be $\zeta \approx 52$, which is nearly eight times greater. This observation leads to confirm that also for the equivalent viscous damping ratio the responses of the two systems are incomparable.

This conclusion reinforces the necessity of defining an alternative to the response spectra such as those used for elastic systems.

V. DISCUSSION AND CONCLUSIONS

The dynamic behavior of rigid free-standing blocks subjected to earthquake ground motions is highly non-linear and sensitive to small perturbations of various parameters. Many difficulties arise in defining reliable response spectra for such systems and these are well known in the literature.

Despite these difficulties, a kind of resonance spectra are derived and presented in this paper in order to highlight to what extent the ground motion details and the system parameters can influence the rocking response.

The first step is focused on the construction of an artificial input aimed at defining such an amplitude resonance as to represent the most likely disadvantageous conditions for the generic block. This is represented by a sequence of instantaneous pulses with interval durations calibrated so as to input constant additional energy at each half-cycle. The rocking response to such input loading is a sequence of natural motions between impacts characterized by a given initial velocity due to each pulse. The simplified equation first given by Housner [1] is used to describe the motion of the block between two subsequent impacts.

The dissipation of the energy due to impact of the block on the ground is represented by the coefficient of restitution which is assumed to be a variable of the problem to account also for other damping effects (e.g. local plastic deformations). The adoption of coefficients of restitution lower than Housner's coefficient has been discussed and interesting relations between this coefficient and the viscous damping ratio are formulated, by means of a comparison with the classical elastic damped model.

Assuming unlimited duration of the pulse sequence and excluding overturning, the results of a parametric analysis are referred to the stabilized phase of motion, for which a threshold for the maximum rotation angle of the block has been defined. The meaningful parameters that influence the stabilized response are the size and the slenderness of the block, the coefficient of restitution and the features of the acceleration pulses representing the resonance input. Each parameter follows a systematic trend as is evident from simple formulations in closed form and each effect on the rocking response has been shown and discussed in this paper.

The results are plotted in proper diagrams that represent resonance spectra, characterized by the fact that the stabilized period strictly depends upon the slenderness ratio and the coefficient of restitution and is quite independent of the size of the block. It means that, for a given coefficient of restitution, the block in condition of resonance can be represented whether by the slenderness or the stabilized period. The latter, therefore, can be considered as the fundamental period, in analogy with elastic systems.

The parametric analysis, herein developed for given intensity and duration of the pulses, highlights that the stability of the block always decreases for: 1) decreasing size of the base; 2) proportional increasing of the slenderness ratio; 3) increasing of the coefficient of restitution. This result is of great importance when considering that generally the stability of the block is extremely sensitive to slight changes in natural accelerograms and that the structural response cannot easily be generalized and parameterized.

The analysis presented in this work may provide a significant step towards reliable response spectra for masonry rigid blocks and open new lines for further theoretical developments and computational applications. At least, it provides additional perspective by which rocking structures can be better understood and assessed.

REFERENCES

[1] G. W. Housner, "The behaviour of inverted pendulum structures during earthquakes," Bulletin of the Seismological Society of America, vol. 53(2), pp. 403-417, 1963.

[2] C. K. Yim, A. K. Chopra and J. Penzien, "Rocking response of rigid blocks to earthquakes," Earthquake Engineering and Structural Dynamics, vol. 8, pp. 565-587, 1980.

[3] M. Aslam, W. G. Godden and D. T. Scalise, "Earthquake rocking response of rigid bodies," Journal of Engineering Mechanics ASCE, vol. 106, pp. 377-392, 1980.

[4] M. J. N. Priestley, R. J. Evison and A. J. Carr, "Seismic response of structures free to rock on their foundations," Bulletin of the New Zealand National Society for Earthquake Engineering, vol. 11(3), pp. 141-150, 1978.

[5] Prestandard and Commentary for the Seismic Rehabilitation of Buildings, FEMA 356, 2000.

[6] N. Makris and D. Konstantinidis, "The rocking spectrum and the limitations of practical design methodologies," Earthquake Engineering and Structural Dynamics, vol. 32, pp. 265-289, 2003.

[7] Y. Ishiyama, "Motions of rigid bodies and criteria for overturning by earthquake excitation," Earthquake Engineering and Structural Dynamics, vol. 10(5), pp. 635-650, 1982.

[8] P. D. Spanos and A. S. Koh, "Rocking of rigid blocks due to harmonic shaking," Journal of Engineering Mechanics ASCE, vol. 110(11), pp. 1627-1642, 1984.

[9] W. K. Tso and C. M. Wong, "Steady state rocking response of rigid blocks. Part 1: Analysis," Earthquake Engineering and Structural Dynamics, vol. 18, pp. 89-106, 1989.

[10] C. M. Wong and W. K. Tso, "Steady state rocking response of rigid blocks. Part 2: Experiment," Earthquake Engineering and Structural Dynamics, vol. 18, pp. 107-120, 1989.

[11] H. W. Shenton III, "Criteria for initiation of slide, rock, and slide-rock rigid-body modes," Journal of Engineering Mechanics ASCE, vol. 122(7), pp. 690-693, 1996.

[12] J. Zhang and N. Makris, "Rocking response of free-standing blocks under cycloidal pulses," Journal of Engineering Mechanics ASCE, vol. 127(5), pp. 473-483, 2001.

[13] C. Casapulla, "Lower and upper bounds in closed form for out-of-plane strength of masonry structures with frictional resistances," in *Proc. SAHC08*, 2008, Bath (UK), vol. 2, p. 1191.

[14] S. Lagomarsino, "Seismic assessment of rocking masonry structures," Bulletin of Earthquake Engineering, vol. 13, pp. 97-128, 2015.

[15] M. J. DeJong, “Amplification of rocking due to horizontal ground motion,” Earthquake Spectra, vol. 28(4), pp. 1405-1421, 2012.
[16] C. Casapulla, P. Jossa and A. Maione, “Rocking motion of a masonry rigid block under seismic actions: a new strategy based on the progressive correction of the resonance response,” Ingegneria Sismica, vol. 27(4), pp. 35-48, 2010.
[17] T. Taniguchi, “Non-linear response analyses of rectangular rigid bodies subjected to horizontal and vertical ground motion,” Earthquake Engineering and Structural Dynamics, vol. 31, pp. 1481-1500, 2002.
[18] I. N. Psycharis, D. Y. Papastamatiou and A. P. Alexandris, “Parametric investigation of the stability of classical columns under harmonic and earthquake excitations,” Earthquake Engineering and Structural Dynamics, vol. 29(8), pp. 1093-1109, 2000.