Approximate analytical solution of MHD flow of an Oldroyd 8-constant fluid in a porous medium

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Abstract----The steady flow in an incompressible, magnetohydrodynamic (MHD) Oldroyd 8-constant fluid in a porous medium with the motion of an infinite plate is investigated. Using modified Darcy's law of an Oldroyd 8-constant fluid, the equations governing the flow are modelled. The resulting nonlinear boundary value problem is solved using the homotopy analysis method (HAM). The obtained approximate analytical solutions clearly satisfy the governing nonlinear equations and all the imposed initial and boundary conditions. The convergence of the HAM solutions for different orders of approximation is demonstrated. For the Newtonian case, the approximate analytical solution via HAM is shown to be in close agreement with the exact solution. Finally, the variations of velocity field with respect to the magnetic field, porosity and non-Newtonian fluid parameters are graphically shown and discussed.

Keywords-Oldroyd 8-constant fluid, Homotopy analysis method, Porous medium, Magnetohydrodynamics (MHD)

I. INTRODUCTION

Several fluids including butter, cosmetics and toiletries, paints, lubricants, certain oils, blood, mud, jams, jellies, shampoo, soaps, soups, marmalades and etc have rheological characteristics and are referred to as non-Newtonian fluids. The rheological properties of all these fluids cannot be explained by using a single constitutive relation between stress and shear rate which is quite different from those viscous fluids [1] and [2]. Such an understanding about non-Newtonian fluids forced researchers to propose more models of non-Newtonian fluids. In general, the classification of non-Newtonian fluid models is given under three categories which are called the differential, the rate and the integral types [3]. In recent years there have been many analytical and numerical studies devoted to the flows of Oldroyd-B (3-constant) fluids which is a subclass of the rate type fluids, see [1-5]. Recently, Hayat et al. [6-7] carried out studies of the flow of an Oldroyd 6-constant fluid in different configurations using the homotopy analysis method (HAM). More recently, Wang and Ellahi [8], Ellahi et al. [9] and [10], Hayat et al. [11], Khan et al. [12] and Sajid et al. [13] investigated the steady state flow of an Oldroyd 8-constant fluid in different configurations using homotopy analysis method (HAM) and some other methods. The Oldroyd 3-constant fluid for a unidirectional steady flow does not exhibit the non-Newtonian property. For this reason some steady flows may be well described by the Oldroyd 8-constant fluid, and thus with this in mind, the chosen model in the present article is of the Oldroyd 8-constant type. The ensuing governing differential equation is non-linear. We solve the non-linear boundary value problems using the homotopy analysis method [14]. This method has already been successfully applied to problems related to non-Newtonian fluids [6, 7, 8, 9, 15 and 16]. To the best of our knowledge, no investigation has been reported so far which discusses the moving flat plate MHD flow of an Oldroyd 8-constant fluid in a porous medium, and thus the objective of the present study is to examine this problem. Constitutive equations of the Oldroyd 8-constant fluid are used and the modified Darcy law is being utilized. The solution to the resulting problem is generated by HAM. Tables show the convergence of the HAM solutions for different orders of approximation and for the Newtonian case the approximate analytical solution via HAM is in close agreement with the exact solution. Graphs are plotted in order to illustrate the variation of the velocity profile with respect to the embedded flow parameters.

II. MATHEMATICAL FORMULATION

A Cartesian coordinate system is chosen by considering an infinite plate at $y=0$. An incompressible fluid occupying the porous space conducts electrically by exerting a uniform magnetic field $B_\circ$, and is applied normal to the plate. The electric field is not taken into consideration and the magnetic Reynolds number is assumed to be small so that the induced magnetic field does not need to be accounted for. The Lorentz force $\boldsymbol{J}\times B_\circ$ under these conditions is equal to $-\sigma B_\circ^2\boldsymbol{V}$. Here $\boldsymbol{J}$ is the current density, $\boldsymbol{V}$ is the velocity field, σ is the electrical conductivity of fluid.

The governing equations are

$$div\boldsymbol{V}=0\ , \tag{1}$$

$$\rho\left[\frac{\partial \boldsymbol{V}}{\partial t}+(\boldsymbol{V}.\nabla)\boldsymbol{V}\right]=-\nabla p+div\ \boldsymbol{S}-\sigma B_\circ^2\boldsymbol{V}+\boldsymbol{R}, \tag{2}$$

in which ρ is the fluid density, p is the pressure, and $\boldsymbol{R}$ is Darcy's resistance.

The extra stress tensor $\boldsymbol{S}$ for an Oldroyd 8-constant fluid satisfies

$$\boldsymbol{T}=-\boldsymbol{I}p+\boldsymbol{S},$$

$$\boldsymbol{S}+\lambda_1\frac{D\boldsymbol{S}}{Dt}+\frac{\lambda_3}{2}\left(\boldsymbol{S}\boldsymbol{A}_1+\boldsymbol{A}_1\boldsymbol{S}\right)\boldsymbol{A}_1+\frac{\lambda_5}{2}\left(tr\boldsymbol{S}\right)+\frac{\lambda_6}{2}\left[\left(tr\boldsymbol{S}\boldsymbol{A}_1\right)\right]\boldsymbol{I}=\mu\left[\boldsymbol{A}_1+\lambda_2\frac{D\boldsymbol{A}_1}{Dt}+\lambda_4\boldsymbol{A}_1^2+\frac{\lambda_7}{2}\left[\left(tr\boldsymbol{A}_1^2\right)\right]\boldsymbol{I}\right], \tag{3}$$

where $\boldsymbol{T}$ is the Cauchy stress tensor, $\boldsymbol{S}$ the extra stress tensor, $\boldsymbol{I}$ the identity tensor, $\boldsymbol{L}$ the velocity gradient, $\boldsymbol{A}=\boldsymbol{L}+\boldsymbol{L}^T$ the first Rivlin – Eriksen tensor, μ the dynamic viscosity of fluid and $\frac{D}{Dt}$ is defined by $\frac{D\boldsymbol{S}}{Dt}=\frac{d\boldsymbol{S}}{dt}-\boldsymbol{L}\boldsymbol{S}-\boldsymbol{S}\boldsymbol{L}^T$, in which $\frac{d}{dt}$ indicates the material derivative.

For steady unidirectional flow, the extra stress tensor and the velocity are denoted respectively, as

$$\boldsymbol{S}\left(y\right)=\begin{bmatrix} S_{xx}\left(y\right) & S_{xy}\left(y\right) & S_{xz}\left(y\right)\\ S_{yx}\left(y\right) & S_{yy}\left(y\right) & S_{yz}\left(y\right)\\ S_{zx}\left(y\right) & S_{zy}\left(y\right) & S_{zz}\left(y\right)\end{bmatrix},\quad \boldsymbol{V}\left(y\right)=\begin{pmatrix} u(y)\\ 0\\ 0\end{pmatrix}. \tag{4}$$

Here $u(y)$ is the velocity in the $x-$ direction.

According to M. Khan et al. [12], the Darcy resistance in Oldroyd 8-constant fluid satisfies the following expression

$$R_x=-\frac{\mu\phi}{k}\left[\left(1+\alpha_1\left(\frac{du}{dy}\right)^2\right)\Big/\left(1+\alpha_2\left(\frac{du}{dy}\right)^2\right)\right]u \tag{5}$$

where

$$\alpha_1=\lambda_1\left(\lambda_4+\lambda_7\right)-\left(\lambda_3+\lambda_5\right)\left(\lambda_4+\lambda_7-\lambda_2\right)-\frac{\lambda_5\lambda_7}{2},$$

$$\alpha_2=\lambda_1\left(\lambda_3+\lambda_6\right)-\left(\lambda_3+\lambda_5\right)\left(\lambda_3+\lambda_6-\lambda_1\right)-\frac{\lambda_5\lambda_6}{2}.$$

ϕ is the porosity and k is the permeability of the porous medium. With the help of Eq. (4), the continuity equation (1) is satisfied identically and the *x*-component of the momentum equation (2) along with Eqs. (3) and (5) for steady flow in the absence of the pressure gradient, and the non-linear model equation is obtained as

$$\frac{d^2u}{dy^2}+\left[\left(3\alpha_1-\alpha_2\right)+\alpha_1\alpha_2\left(\frac{du}{dy}\right)^2\right]\left(\frac{du}{dy}\right)^2\frac{d^2u}{dy^2}$$
$$-\left[N^2\left(1+\alpha_2\left(\frac{du}{dy}\right)^2\right)^2+\frac{\phi}{k}\left(1+\alpha_1\left(\frac{du}{dy}\right)^2\right)\left(1+\alpha_2\left(\frac{du}{dy}\right)^2\right)\right]u=0 \tag{6}$$

where

$$N=\sqrt{\frac{\sigma}{\mu}}B_\circ .$$

The last fifth, non-linear, term in equation (6) represents the porous effects in the model equation. In order to solve this equation, the proficient method HAM is resorted, the corresponding results with the porous effects are taken into consideration, and the influence of pertinent parameters on the fluid motion is graphically underlined.

III. FLOW ANALYSIS

The steady flow of an incompressible, electrically conducting Oldroyd 8-constant fluid occupying the space $y>0$ is considered. The fluid is bounded by an infinite non conducting rigid plate at $y=0$. A uniform magnetic field $B_\circ$ is applied normal to the plate. The flow is maintained due to the sudden motion of the plate, and there is no flow far away from the plate. The governing nonlinear differential equation is ODE (6) and the boundary conditions are

$$u=U_0 \text{ for } y=0,$$
$$u\to 0 \text{ as } y\to\infty, \tag{7}$$

where U_0 is the constant plate velocity.

Introducing the following dimensionless variables

$$\zeta=\frac{U_0}{\nu}y,\ f=\frac{u}{U_0},\ M=\frac{N\nu}{U_0},\ \frac{1}{K}=\frac{\phi\nu}{kU_0},\alpha=\frac{\alpha_1U_0^4}{\nu},\ \delta=\frac{\alpha_2U_0^4}{\nu} \tag{8}$$

where $\nu=\dfrac{\mu}{\rho}$ is the kinematic viscosity.

The problem (6) with conditions (7) now become

$$\frac{d^2f}{d\zeta^2}+\left[\left(3\alpha-\delta\right)+\alpha\delta\left(\frac{df}{d\zeta}\right)^2\right]\left(\frac{df}{d\zeta}\right)^2\frac{d^2f}{d\zeta^2}$$
$$-\left[M^2\left(1+\delta\left(\frac{df}{d\zeta}\right)^2\right)^2+\frac{1}{K}\left(1+\alpha\left(\frac{df}{d\zeta}\right)^2\right)\left(1+\delta\left(\frac{df}{d\zeta}\right)^2\right)\right]f=0 \tag{9}$$

with $f(0)=1$ and $f(\infty)=0.$ (10)

IV. ESSENTIAL IDEAS OF HAM

Consider

$$\mathcal{N}\left[u(r,t)\right]=0, \tag{11}$$

which is a nonlinear equation in a general form, where $\mathcal{N}$ indicates a nonlinear operator, $u(r,t)$ implies a unknown function. By using homotopy analysis method, the zeroth-order deformation equation is constructed as

$$(1-q)\ell\left[\phi(r,t;q)-u_0(r,t)\right]=q\hbar\mathcal{H}(r,t)\mathcal{N}\left[\phi(r,t;q)\right], \tag{12}$$

where $u_0(r,t)$ denotes an initial guess of the exact solution $u(r,t)$, $\mathcal{H}(\mathrm{r},\mathrm{t}) \neq 0$ an auxiliary function, ℓ an auxiliary linear operator, $\hbar \neq 0$ an auxiliary parameter, and $q \in [0,1]$ is an embedding parameter.

It should be noted, that the auxiliary parameter $\hbar$ which is being featured in HAM can be chosen freely. Obviously, when $q = 0,1$ Eq. (12) holds for

$$\phi(r,t;0) = u_0(r,t),\ \phi(r,t;1) = u(r,t)$$

respectively. Then as long as q increases from 0 to 1, the solution $\phi(r,t;q)$ varies from the initial guess $u_0(r,t)$ to the exact solution $u(r,t)$.

By using Taylor theorem, Liao [17] has expanded $\phi(r,t;q)$ in a power series of q as follows

$$\phi(r,t;q) = \phi(r,t;0) + \sum_{m=1}^{\infty} u_m(r,t) q^m \tag{13}$$

where

$$u_m(r,t) = \frac{1}{m!} \frac{\partial^m \phi(r,t;q)}{\partial q^m}\Big|_{q=0} \tag{14}$$

The convergence of the series (13) depends upon the auxiliary function $\mathcal{H}(r,t)$, auxiliary parameter $\hbar$, auxiliary linear operator ℓ and initial guess $u_0(r,t)$. If these are selected properly, the series (13) is convergence at $q = 1$, and one has

$$u(r,t) = u_0(r,t) + \sum_{m=1}^{\infty} u_m(r,t) \tag{15}$$

Based on Eq. (14), the governing equation can be derived from the zeroth-order deformation equation (12) and we can define the vector

$$\overrightarrow{u_n(r,t)} = \{u_0(r,t), u_1(r,t), \ldots, u_n(r,t)\}.$$

Differentiating m-times of zeroth-order deformation equation (12) with respect to q and dividing them by $m!$ and also setting $q = 0$, the result will be so-called mth-order deformation equation

$$\ell\left[u_m(r,t) - \chi_m u_{m-1}(r,t)\right] = \hbar \mathcal{H}(r,t) \mathcal{R}_m(u_{m-1}, r, t) \tag{16}$$

where

$$\chi_m = \begin{cases} 0 & , m \le 1 \\ 1 & , m > 1 \end{cases} \tag{17}$$

$$\mathcal{R}_m(u_{m-1}, r, t) = \frac{1}{(m-1)!} \left\{ \frac{\partial^{m-1}}{\partial q^{m-1}} \mathcal{N}\left[\sum_{m=0}^{\infty} u_m(r,t) q^m \right] \right\}\Big|_{q=0} \tag{18}$$

THEOREM **3.1** (Liao [17])**:**

The series (15) is convergent to the exact solution of (11) as long as the series is convergent.

V. SOLUTION OF THE PROBLEM

Now to solve the non-linear ordinary differential equation (9) subject to boundary conditions (10), the method HAM is applied to give an approximate, uniformly valid and analytic solution. For the HAM solution, the initial guess function is chosen to be in the form

$$f_0(\zeta) = e^{-\zeta} \tag{19}$$

and

$$\mathcal{L} = f'' - f \tag{20}$$

as the auxiliary linear operator satisfying

$$\mathcal{L}\left(C_1 e^{\zeta} + C_2 e^{-\zeta}\right) = 0, \tag{21}$$

where C_1 and C_2 are arbitrary constants.

It is very significant that one has great freedom to choose auxiliary objects in HAM in accordance to the rule of its solution expression. Moreover, any value of the initial operator may be selected which yields rapid convergence in the given domain. In the present problem, the flow domain is semi-infinite and therefore having the boundary condition at infinity, an initial operator of the form (20) which satisfies Eq. (21) is chosen. If $q \in [0,1]$ is the embedding parameter and $\hbar$ is the auxiliary nonzero parameter then we have:

Zeroth-order deformation

$$(1-q)\mathcal{L}\left[f(\zeta,q) - f_0(\zeta)\right] = q\hbar\mathcal{N}\left[f(\zeta,q)\right], \tag{22}$$

$$f(0,q) = 1,\ f(\infty,q) = 0, \tag{23}$$

$$\begin{aligned}\mathcal{N}\left[f(\zeta,q)\right] &= \frac{d^2 f(\zeta,q)}{d\zeta^2} + (3\alpha-\delta)\left(\frac{df}{d\zeta}\right)^2 \frac{d^2 f}{d\zeta^2} + \alpha\delta\left(\frac{df}{d\zeta}\right)^4 \frac{d^2 f}{d\zeta^2} \\ &-M^2 f(\zeta,q)\left(1+\delta^2\left(\frac{df}{d\zeta}\right)^4 + 2\delta\left(\frac{df}{d\zeta}\right)^2\right) - \frac{1}{K} f(\zeta,q)\left(1+(\alpha+\delta)\left(\frac{df}{d\zeta}\right)^2 + \delta\alpha\left(\frac{df}{d\zeta}\right)^4\right).\end{aligned} \tag{24}$$

Mth-order deformation problem

$$\mathcal{L}\left[f_m(\zeta) - \chi_m f_{m-1}(\zeta)\right] = \hbar\mathcal{R}_m(\zeta), \tag{25}$$

$$f_m(0) = 0, \quad f_m(\infty) \to 0, \tag{26}$$

where

$$\begin{aligned}\mathcal{R}_m(\zeta) &= \frac{d^2 f_{m-1}}{d\zeta^2} - \left(M^2 + \frac{1}{K} \right) f_{m-1} + \sum_{k=0}^{m-1}\sum_{i=0}^{k} \frac{df_{m-1-k}}{d\zeta}\frac{df_{k-i}}{d\zeta}\left[(3\alpha-\delta)\left(\frac{d^2 f_i}{d\zeta^2}\right) - \left(2\delta M^2 + \frac{1}{K}(\alpha+\delta)\right) f_i \right] \\ &+ \sum_{k=0}^{m-1}\sum_{i=0}^{k}\sum_{j=0}^{i}\sum_{l=0}^{j} \frac{df_{m-1-k}}{d\zeta}\frac{df_{k-i}}{d\zeta}\frac{df_{i-j}}{d\zeta}\frac{df_{j-l}}{d\zeta}\left[\alpha\delta\frac{d^2 f_l}{d\zeta^2} - \left(\delta^2 M^2 + \alpha\delta\frac{1}{K} \right) f_l \right]\end{aligned} \tag{27}$$

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases} \tag{28}$$

MATHEMATICA is used to solve the set of linear equations (25) with conditions (26). It is found that the series solution form is given by

$$\begin{aligned} f(\zeta) &= e^{-\zeta} + \frac{1}{24}e^{-5\zeta}\left(-12\zeta e^{-4\zeta} + 9\alpha e^{2\zeta} - 9\alpha e^{4\zeta} + 12K\zeta e^{4\zeta} - 3\alpha K e^{2\zeta} + 3\alpha K e^{4\zeta} + \ldots \right. \\ &\left. + 6\delta M e^{4\zeta} - 6\delta M e^{2\zeta} \right)\hbar + \frac{1}{5760}e^{-9\zeta}\left(-720\delta K\hbar e^{6\zeta} + 720\delta K\hbar e^{8\zeta} - 240\delta K\hbar\alpha e^{4\zeta} \right. \\ &\left. + 240\delta K\alpha\hbar e^{8\zeta} + \ldots\ldots \right)\end{aligned} \tag{29}$$

The approximate analytical solution given by (29) contains the auxiliary parameter $\hbar$, which influences the convergence region and rate of approximation for the HAM solution. In Fig.1 the $\hbar$ - curve is plotted for $f(\zeta)$ when $\zeta = 5, \alpha = \delta = M = K = 1$ at 3th-order approximation.

Table 1 shows the convergence of the HAM solutions for different orders of approximation. This implies that the 3rd order is the appropriate order of approximation to be chosen.

Table1

Convergence of the HAM solutions for different orders of approximation when $M = 3$ and $\alpha = \delta = K = 1$

Table 1

Order of approximation	$f(0)$	$-f'(0)$	$f''(0)$
1	1	1	1
2	1	1.245	1.84
3	1	1.48351	2.8564
5	1	1.48351	2.8564
8	1	1.48351	2.8564

Fig. 1 clearly elucidates that the range for the admissible values of $\hbar$ is $-0.5 \le \hbar \le 0.0$. These calculations indicate that the series solution as given in Eq. (29) converges in the whole region of ζ when $\hbar = -0.2$.

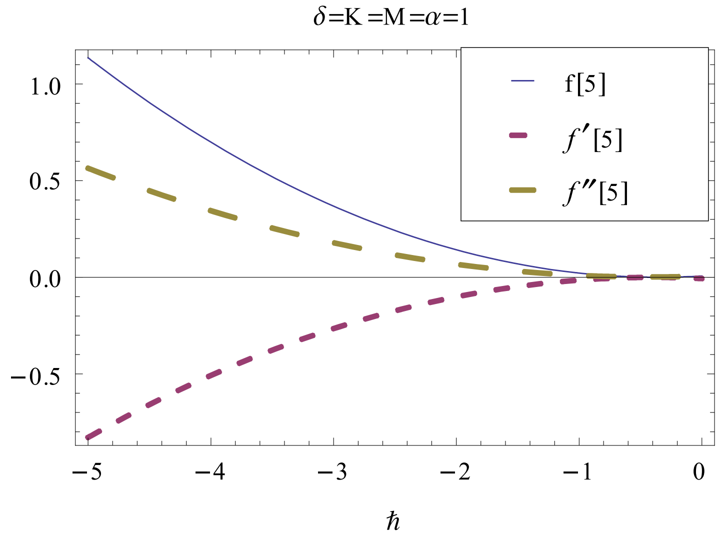

Fig.1. $\hbar$ -curve

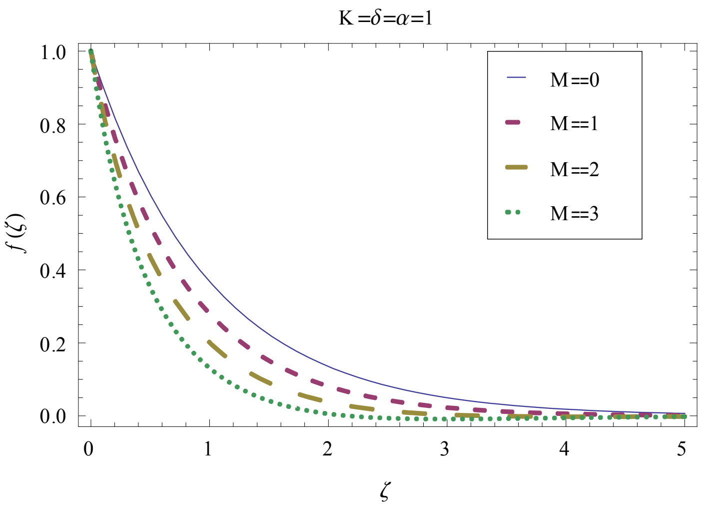

Fig.2. Velocity profile for MHD parameter M.

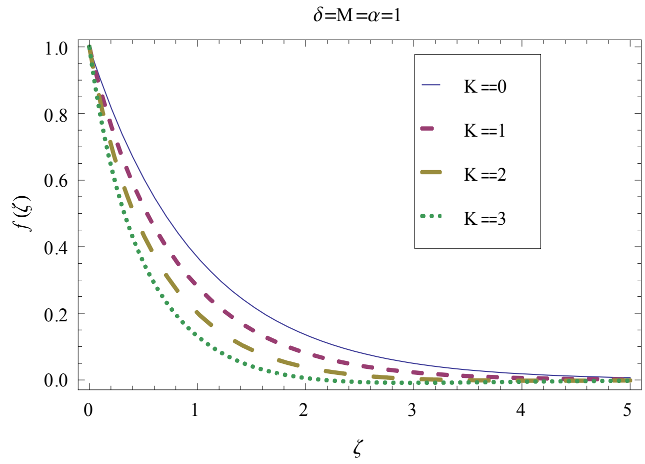

Fig.3. Velocity profile for porous parameter when K.

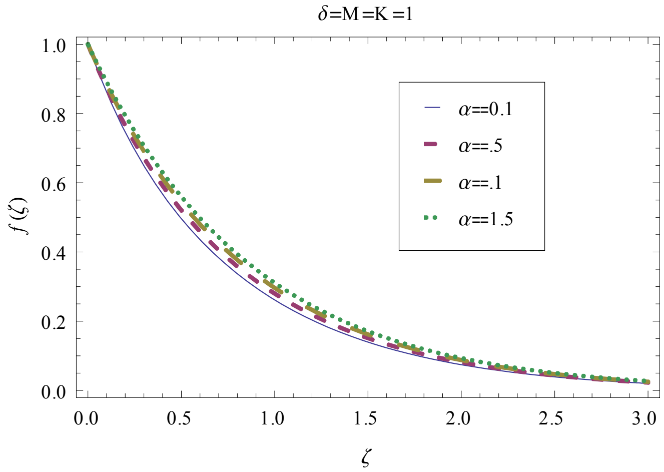

Fig.4. Velocity profile for parameter α .

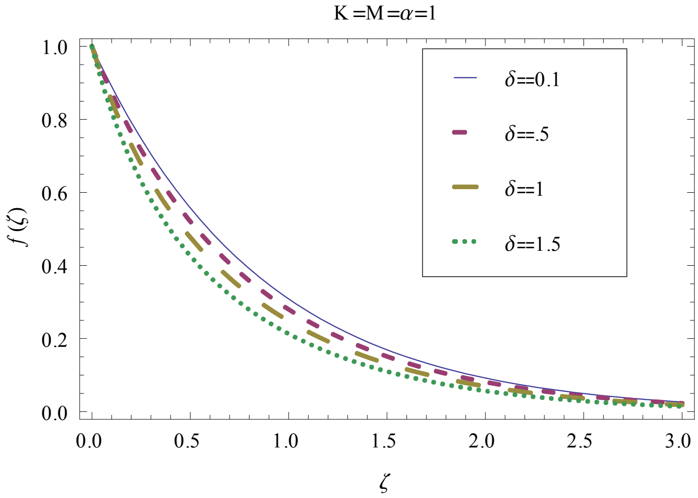

Fig.5. Velocity profile for parameter δ .

VI. RESULT AND DISCUSSION

In this section, the graphical illustrations of the velocity profiles as shown in Figures 2-5 are described. These graphs have been determined for the MHD flow of the Oldroyd–8 constant fluid in a porous medium over a moving flat plate. The emerging parameters here are

(1) M is the magnetic field parameter,

(2) K is the porous parameter,

(3) α & δ are the material constant parameters.

In order to illustrate the role of these parameters on the velocity profile of $f(\zeta)$, the Figs. 2 - 5 have been displayed.

Fig. 2 is prepared to see the effects of applied magnetic field (Hartman number) M on the velocity profile. Keeping $\alpha, \delta, K, \hbar$ fixed and varying M, it is noted that the velocity profile decreases by increasing the magnetic field parameter M. Clearly, we observe that with increasing the values of M, the velocity profile of $f(\zeta)$ decreases, in fact this is because of the effects of the transverse magnetic field on the electrically conducting fluid which gives rise to a resistive type Lorentz force which tends to slow down the motion of the fluid.

Very interesting situation is observed from the results in Fig. 3. By increasing the porous parameter K whiles the other parameters $M, \alpha, \delta, \hbar$ are fixed, would lead to a decrease in the velocity profile. This is physically due to the fact that by increasing the values of K the medium would appear to be some less porous and thus this increases the friction forces, and would then reduce the flow of the fluid.

Fig. 4 shows the effects of the material constant parameter α on the velocity profile when $M, K, \delta, \hbar$ are fixed. It is of interest to notice that by increasing the parameter α, this would lead to an increase in the velocity profile (this is much related to decrease in the boundary layer thickness). This is in fact true since by increasing the values of α would then reduce the friction forces, and, thus, assists the flow of the fluid considerably; and hence the fluid moves with greater velocity.

Fig. 5 shows the effects of the other material constant parameter δ on the velocity profile when $M, K, \alpha, \hbar$ are fixed. It is worth noticing that by increasing the parameter δ would lead to a decrease in the velocity profile (this is much related to increase in the boundary layer thickness). This is because of the fact that increasing the values of δ would increase the friction forces, and, thus, slow down the motion of the fluid.

Table 2 shows that for the Newtonian case, the approximate analytical solution via HAM is in close agreement with the exact solution. Thus this verifies the form of the HAM solution for this problem.

Table2

Illustrating the variation of $-f'(0)$ when $M = .1, K = 1$ and $\alpha = \delta = 0$ i.e. Newtonian case using HAM and exact solution (Faisal et al. [5])

Table 2

z	$-f'(z)$	
	HAM	**Exact [5]**
0	1.00675	1.04881
.1	0.910332	0.944381
.2	0.823149	0.85035
.3	0.744316	0.765683
.4	0.673032	0.689445
.5	0.608575	0.620798
.6	0.55029	0.558986
.7	0.497588	0.503329
.8	0.449933	0.453213
.9	0.406842	0.408088
1.00	0.367877	0.367455

VII. CONCLUDING REMARKS

In this paper, we have modelled the governing equation for steady and magnetohydrodynamic flow of an Oldroyd 8-constant fluid over a plate in a porous medium and determined the corresponding velocity field. The HAM is used to solve the governing non-linear second-order differential equation. The obtained solution indicates the effects of material constant parameters, porous parameter and the magnetic field. It is shown that an increase in the applied magnetic field leads to a decrease in the velocity due to the resistive Lorentz force. From the presented analysis, the main observations are described as follows:

i. The behaviours of α and δ on the velocity $f(\zeta)$ based on the graphs are quantitatively opposite in nature.
ii. The flow characteristics subject to M and K on the velocity components based on the graphs leads to the same effects.
iii. The results corresponding to viscous fluid can be obtained by choosing $\alpha = \delta = 0$.
iv. The convergence of the HAM solutions for different orders of approximation is demonstrated.
v. For the Newtonian case, the approximate analytical solution via HAM is shown to be in close agreement with the exact solution.

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