Fractional Order Modelling Using State Space Theory

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***Abstract* - There are various fractional order systems existing. This paper deals with the modelling of fractional order systems using an old and unique model structure i.e. state space model. The fractional order process system can be mathematically modelled by state space model. Simulation results validated that the fractional order model using state space is better as compared to other models such as first order with delay.**

Keywords- Fractional Calculus, State Space Model, Fractional Control

I. INTRODUCTION

The fractional calculus is more than three centuries old like conventional calculus, but not very popular amongst science and engineering community [1]. It has numerous applications in the field of control systems and process modelling. Mathematical model is used for optimal control, process optimization, soft sensor, prediction and so on. Nowadays, use of mathematical models is increasing because of model based control and model based design in chemical process systems like heat exchangers, boilers etc.

In fractional calculus, the two most commonly used definitions for the general fractional integro-differential operators are the Grünwald-Letnikov (GL) and Riemann-Liouville (RL).

The GL definition is given by

$$aD_t^q f(t) = \lim_{h\to 0} h^{-\alpha} \sum_{j=0}^{[t-q/h]} (-1)^j \binom{q}{j} f(t-jh)$$

Where, $[.]$ implies the integral part.

And RL Definition is given by

$$aD_t^q f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dt} \int_a^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} d\tau$$

Where $n-1 < q < n$ and $\Gamma(.)$ is the Euler's gamma function. [3]

Also, M. Caputo, also given definition of fractional Integration as

$$aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau$$

The objective of this work is to design and model a fractional order system. There are many natural fractional order systems present today. For capturing the dynamics of fractional order system, an old and unique model structure named state space model has been used in this work. Simulink has been utilised to gather data of fractional order system. Using state space based model identification method, dynamics of fractional order system have been captured into the state space model.

This paper is organized in different sections. The first section covers an introduction of the work. Second section briefly reviews literature of fractional calculus its modelling. The proposed method for fractional order modelling is described in section 3 with experimental work for simulated system followed by conclusions and references.

II. FRACTIONAL ORDER MODELLING

Modelling of fractional order system is one of the most challenging tasks. By using developed models, we can design advance control system, soft sensor, operational optimization, fault diagnosis, forecasting and soft sensors. [14]

Fractional order system has following general mathematical form

$$G(s) = \frac{k}{s^{a_1} + bs^{a_2} + cs^{a_3} + d}$$

Where, a1, a2 and a3 are real numbers and b,c,d are integer constants.

Using sub space model, this model is identified in state space form as follows

State-space model: x(t+Ts) = A x(t) + B u(t) + K e(t)

y(t) = C x(t) + D u(t) + e(t)

Where x(t) is the state of the system,

u(t) is called the input vector,

y(t) is called the output vector,

e(t) is called the error vector,

A is called state matrix,

B is called input matrix,

C is called output matrix and

D is called feed forward matrix

Generally, modelling of any system comprises of the following steps. First step is to set up the experiment. Then the system is excited by step input and data is collected from the experimental runs. In this work, the state space model has been selected and parameters of model have been estimated by an estimation theory. Also, the developed model has been validated against experimentally generated data.

II. EXPERIMENTAL SET UP AND RESULT

The experimental set up for fractional order system (to collect data for modelling) is shown in figure 1. The selected fractional order model is

$$G(s) = \frac{1}{s^{2.5} + 1}$$

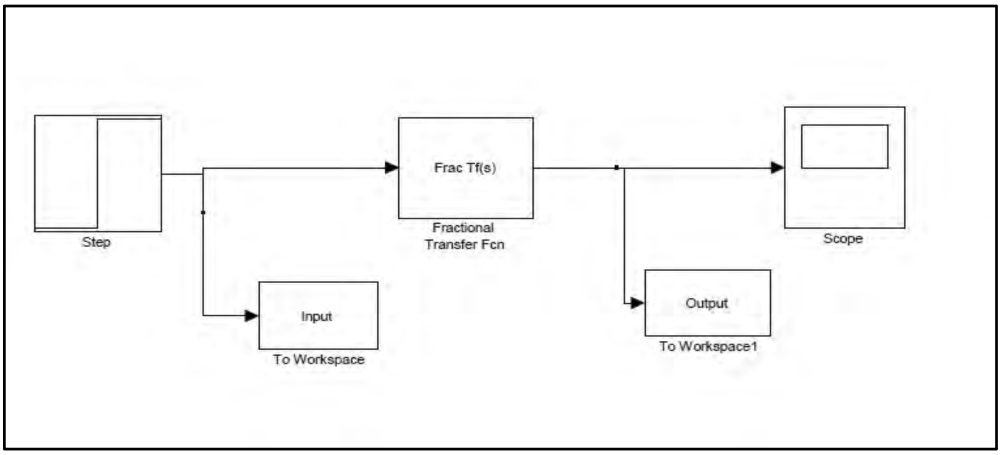

Figure: 1 Experimental Set up for Data Collection for Fractional Order System

Figure 2 and 3 show plots of time responses and frequency responses of the collected data, respectively.

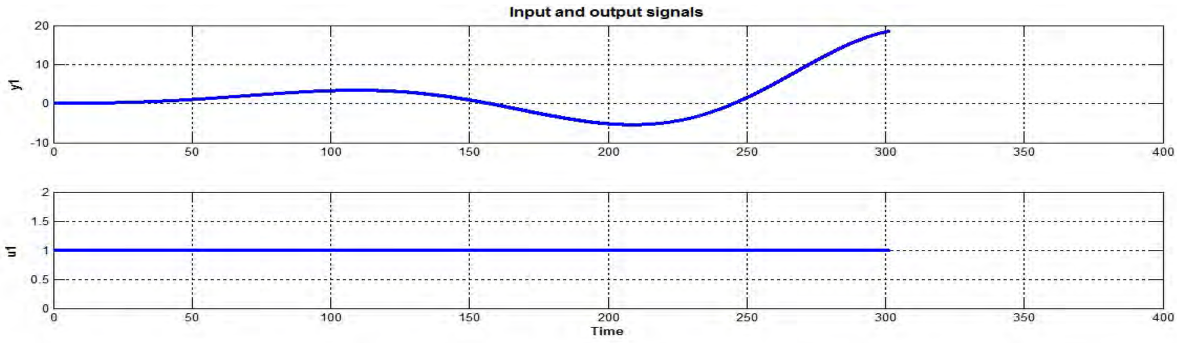

Figure:2 Time plot of Input and Output for given Fractional Order System

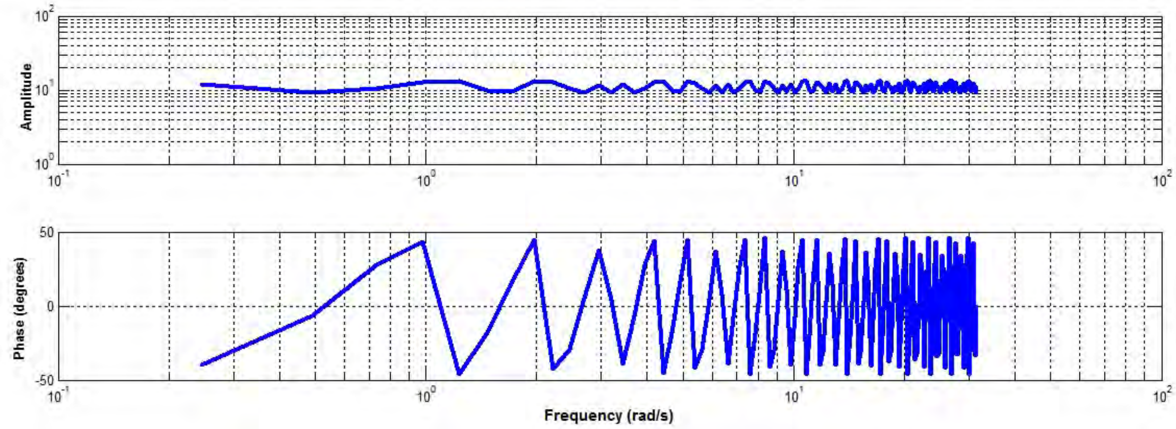

Figure:3 Frequency plot of Input and Output for given Fractional Order System

The following model structure has been selected for capturing dynamics of fractional order system

State-space model: $x(t+Ts) = A\,x(t) + B\,u(t) + K\,e(t)$

$$y(t) = C\,x(t) + D\,u(t) + e(t)$$

Where, $x(t)$ is called state vector,

$y(t)$ is called output vector.

The state-space model has been estimated using a sub-space method. This model fitted almost 96 percent of the experimental data.

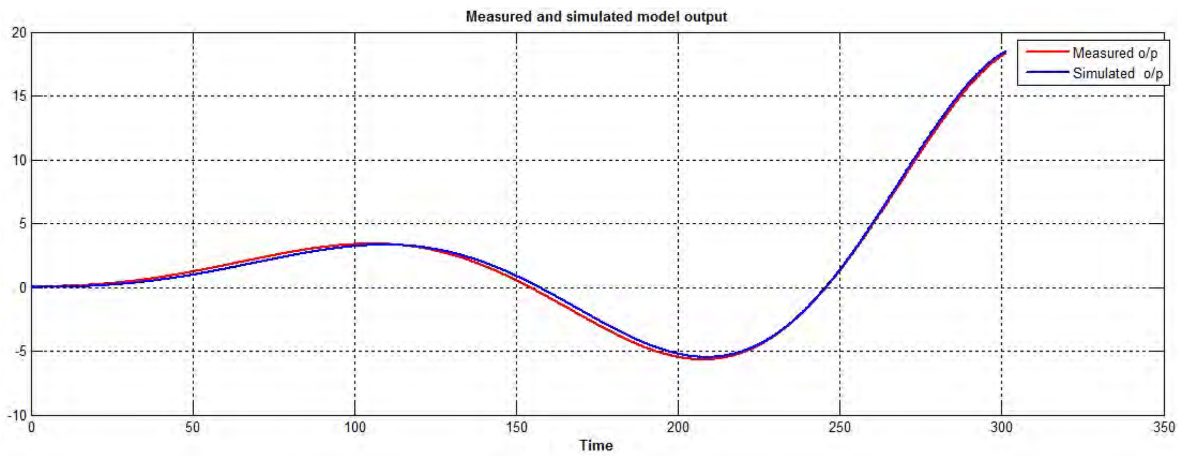

Figure 4: Measured and simulated model outputs for State Space Model

Modelling of fractional order using First Order Delay System model:

Figure 5 shows measured and simulated model outputs for First Order Delay System. The obtained model is:

Process model with transfer function

$$G(S) = \frac{K_p}{1 + T_{p1} * s} * e^{-T_d * s}$$

Kp = 4.1061e+008, Tp1 = 2.1323e+010 and Td = 3

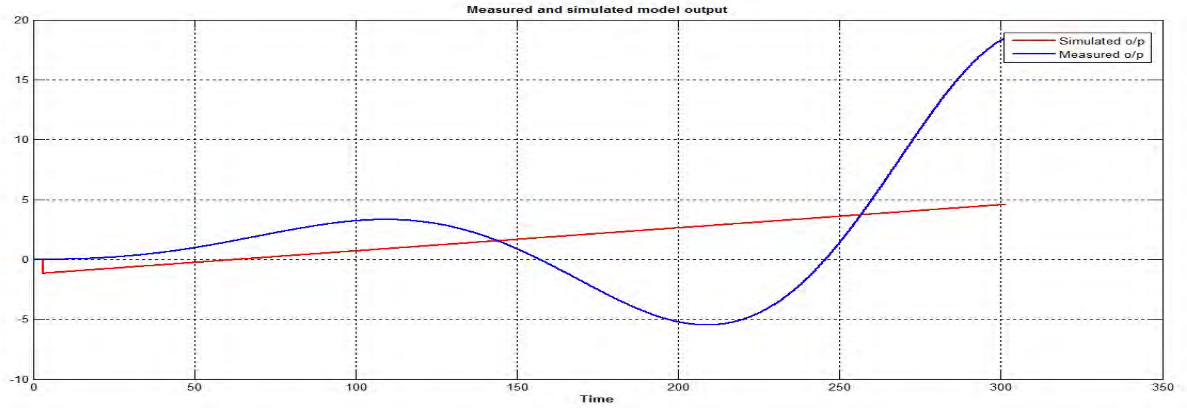

Figure:5 Measured and simulated model outputs for First Order Delay System

The results of different modelling methods are compared (Table 1). It is clear from the comparison that for state space model FPE (Final Predication Error) is very less as compared to first order model.

Table 1 Comparing Results

Sr. No	**Model**	**Loss Function**	**FPE**
1	State Space Model	5.45958e-006	5.51756e-006
2	First Order Delay System	25.5362	25.5871

IV. CONCLUSION

In this paper, it has been shown that the state space based model gives good results. It matches upto 96% of the dynamics of fractional order system. The fractional order system dynamics have been successfully captured into the state space model. This achievement enables an appropriate design of model based control systems for fractional order system. The comparison of results shows a definite advantage of state space model over the standard first order with delay system.

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