STRONG NONSPLIT X-DOMINATING SET OF BIPARTITE GRAPHS

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***Abstract*— Let G be a bipartite graph. A X-dominating set D of X of G is a strong nonsplit X-dominating set of G if every vertex in X-D is X-adjacent to all other vertices in X-D. The strong nonsplit X-domination number of a graph G, denoted by $\gamma_{snsX}(G)$ is the minimum cardinality of a strong nonsplit X-dominating set. We find the bounds for strong nonsplit X-dominating set and give its bipartite version.**

Keyword- Strong nonsplit X-dominating set, X-dominating set, X-clique, strong nonsplit dominating set.

I. INTRODUCTION

Let G be a simple graph. The bipartite theory of graphs was formulated by Hedetniemi and Laskar in [1,2] which states that for any problem, say P, on an arbitrary graph G, there is a corresponding problem Q on a bipartite graph G1, such that a solution for Q provides a solution for P. The parameter called X-dominating set and Y-dominating set was introduced in [1,2] and was further studied in [5]. The bipartite version of irredundant set, domination in complement of a graph was discussed in [6,7]. In this paper, we define strong nonsplit X-dominating set and give its bipartite version.

II. PRELIMINARIES

Let G = (V,E) be a graph. The number of vertices of G we denote by n. By the neighbourhood of a vertex v of G we mean the set $N_G(v) = \{u : u \text{ and } v \text{ are adjacent}\}$. We say that a vertex is isolated if it has no neighbour, while it is universal if it is adjacent to all other vertices. The degree of a vertex v, denoted by $d_G(v)$, is the cardinality of its neighbourhood. Let $\delta(G)$ mean the minimum degree among all vertices of G. We say that a subset of V(G) is independent if there is no edge between any two vertices of this set. The independence number of a graph G, denoted by $\beta(G)$, is the maximum cardinality of an independent subset of the set of vertices of G. The clique number of G, denoted by $\omega(G)$, is the number of vertices of largest complete graph which is a subgraph of G.

A vertex of a graph is said to dominate itself and all its neighbours. A subset D of V(G) is a dominating set of G if every vertex of G is dominated by at least one vertex of D. The domination number of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G. Many types of domination parameters were defined and the reader is referred to a comprehensive survey of domination in graphs, see [3,4].

Let G be a bipartite graph. Two vertices u and v in X are X-adjacent if they are adjacent to a common vertex y in Y. A subset S of X is a X-dominating set if every vertex in X-S is X-adjacent to a vertex of S. The minimum cardinality of a X-dominating set is called the X-domination number of a graph G and is denoted by $\gamma_X(G)$. Two vertices are said to be X-independent if they are not X-adjacent. A subset D of X is called a X-independent set if any two vertices in D are not X-adjacent. The maximum cardinality of a X-independent set is c alled the X-independence number of G and is denoted by $\beta_X(G)$. A subset S of X is a X-clique if any two vertices of S are X-adjacent. The maximum cardinality of a X-clique denoted by $\omega_X(G)$ is called the X-clique number of G.

III. STRONG NONSPLIT X-DOMINATION NUMBER OF A BIPARTITE GRAPH

Definition 1: A X-dominating set of G is said to be a strong nonsplit X-dominating set of G if every vertex in X-D is X-adjacent to all other vertices in X-D. The strong nonsplit X-domination number of a graph G, denoted by $\gamma_{snsX}(G)$ is the minimum cardinality of a strong nonsplit X-dominating set.

Remark 2: Let G be a bipartite graph with at least one non Y-isolat say x. Then X-{x} is a strong nonsplit X-dominating set of G. Hence, every bipartite graph with at least one non Y-isolate has a strong nonsplit X-dominating set of G.

Theorem 3: Let G be a bipartite graph with p>2 and there exists vertices u,v,w which are mutually X-adjacent. Then $\gamma_{snsX}(G) \leq p-2$.

Proof: By hypothesis, there exists vertices u,v,w which are mutually X-adjacent. Then, X-{u,v} is a strong nonsplit X-dominating set of G. Therefore, $\gamma_{snsX}(G) \leq p-2$.

Theorem 4: For any connected bipartite graph G, $\beta_X(G) \leq \gamma_{snsX}(G)$.

Proof: Let D be a $\gamma_{snsX}-$set of G. Then any two vertices in X-D are X-adjacent. Moreover every vertex in X-D is X-adjacent to a vertex of D. Therefore, $\beta_X(G) \leq |D| = \gamma_{snsX}(G)$. The bound is attained in $K_{m,n}$.

Notation: Let S_p be a bipartite graph (X, Y, E), $|X| = p; |Y| = p-1$ with a vertex in X, X-adjacent to all other vertices of X through different y in Y and all vertices in X-{x} are end vertices.

We now compute the strong nonsplit X-domination number of standard graphs.

Theorem 5: (i) For the complete bipartite graph $K_{m,n}$ $\gamma_{snsX}(K_{m,n}) = 1$.

(ii) For the cycle C_{2n}, $\gamma_{snsX}(C_{2n}) = \begin{cases} |X|-2 & if \ \ n \neq 2 \\ 1 & if \ \ n = 2,3 \end{cases}$

(iii) For the graph S_p, $\gamma_{snsX}(S_P) = p-1$.

Proof: (i) We know that $1 = \beta_X(K_{m,n}) \leq \gamma_{snsX}(K_{m,n})$. Any vertex in X is a strong nonsplit X-dominating set. Therefore, $\gamma_{snsX}(K_{m,n}) \leq 1$. Hence, $\gamma_{snsX}(K_{m,n}) = 1$.

(ii) The graph C_4 is $K_{2,2}$. Hence, $\gamma_{snsX}(K_{2,2}) = 1$. In the case of n=3, any vertex of X is a strong nonsplit X-dominating set and $1 = \beta_X(C_6) \leq \gamma_{snsX}(C_6)$. Therefore, $\gamma_{snsX}(C_6) = 1$. For n>3, let u,v and w be mutually X-adjacent vertices in X(G). Then, $X(C_{2n}) - \{u,v\}$ is a strong nonsplit X-dominating set. Hence, $\gamma_{snsX}(C_{2n}) \leq |X|-2$. Clearly, the above set is minimum strong nonsplit X-dominating set. Hence, $\gamma_{snsX}(C_{2n}) = |X|-2$.

(iii) We have $p-1 = \beta_X(S_p) \leq \gamma_{snsX}(S_p)$. The set X-{x} where x is the vertex X-adjacent to all other vertices of X is a strong nonsplit X-dominating set. Therefore, $\gamma_{snsX}(S_p) \leq p-1$. Hence, $\gamma_{snsX}(S_P) = p-1$.

Definition 6: A strong nonsplit X-dominating set D of a graph G is a minimal strong nonsplit X-dominating set if no proper subset of D is a nonsplit X-dominating set of G.

We now characterize minimal strong nonsplit X-dominating set of graph G.

Theorem 7: A strong nonsplit X-dominating set D of G is minimal if and only if for all $v \in D$, one of the following conditions hold:

(i) The vertex v is an Y-isolate of D.

(ii) There exists a vertex u in X-D such that u is Y-private neighbor of v.

(iii) There exists a vertex w in X-D such that w is not X-adjacent to v.

Proof: Let D be a minimal strong nonsplit X-dominating set. Let $v \in D$, then $D-\{v\}$ is not a strong nonsplit X-dominating set. Either there exists $w \in X-(D-\{v\})$ which is not X-adjacent to $v \in D$ or vertices in $X-(D-\{v\})$ are not complete.

Case (i): There exists $w \in X-(D-\{v\})$ which is not X-adjacent to $v \in D$ then either $v = w$ in which case v is an Y-isolate of D which is (i) or $w \in X-D$. If w is not X-adjacent with any vertex in D then w is a Y-private neighbor of v which is (ii).

Case (ii): Vertices in $X-(D-\{v\})$ are not X-complete. Equivalently there is a vertex $w \in X-D$ which is not X-adjacent to v which is (iii).

Conversely, let for some $v \in D$ some of the three conditions hold. Then $D-\{v\}$ is a X-dominating set of G such that $(X-D)\cup\{v\}$ is X-complete. Therefore, $D-\{v\}$ is a strong nonsplit X-dominating set of G. That is D is not a minimal strong nonsplit X-dominating set of G.

The complement of a minimal strong nonsplit dominating set is not a strong nonsplit dominating set. The complement of a minimal strong nonsplit dominating set is also a minimal strong nonsplit dominating set if some conditions are imposed as given in the following theorem.

Theorem 8: Let G be a graph with $\Delta_Y(G) \le p-2$. Let D be a strong nonsplit X-dominating set of G such that $\langle D \rangle$ is a X-clique and $|D| \le \delta_Y(G)$. Then (i) D is a minimal nonsplit X-dominating set. (ii) The set X-D is also a minimal strong nonsplit X-dominating set of G.

Proof: Since $\Delta_Y(G) \le p-2$, for every v in D, there exists w in X-D such that v and w are not X-adjacent. Hence, D is a minimal nonsplit X-dominating set. Since $|D| \le \delta_Y(G)$, every vertex in D is X-adjacent to some vertex in X-D. Since $\langle D \rangle$ is a X-clique, the set X-D is a strong nonsplit X-dominating set of G. Also by the above theorem, we have X-D is a minimal strong nonsplit X-dominating set of G.

We now give the lower and upper bounds of strong nonsplit X-domination number of a graph G.

Theorem 9: For any bipartite graph G, $p-\omega_X(G) \le \gamma_{snsX}(G) \le p-\omega_X(G)+1$.

Proof: Let D be a $\gamma_{snsX}-$set. Then X-D is a X-clique. Therefore, $\omega_X(G) \ge |X-D| = p-\gamma_{snsX}(G)$. Therefore, $p-\omega_X(G) \le \gamma_{snsX}(G)$.

Let S be a X-clique set of order $\omega_X(G)$. Then, $(X-S)\cup\{w\}, w \in S$ is a strong nonsplit X-dominating set. Hence, $\gamma_{snsX}(G) \le |X-S|+1 = p-\omega_X(G)+1$.

Theorem 10: Let G be a connected bipartite graph with $\omega_X(G) \ge \delta_Y(G)$. Then $\gamma_{snsX}(G) \le p-\delta_Y(G)$ and the bound is attained if and only if one of the following conditions is satisfied (i) $\omega_X(G) = \delta_Y(G)$ (ii) $\omega_X(G) = \delta_Y(G)+1$ and every ω_X-set S of X contains a vertex not X-adjacent to any vertex of X-S.

Proof: Suppose $\omega_X(G) \ge \delta_Y(G)+1$. Then, $\gamma_{snsX}(G) \le p-\omega_X(G)+1 \le p-\delta_Y-1+1 = p-\delta_Y(G)$. Let $\omega_X(G) = \delta_Y(G)$. Let S be a ω_X-set of G with $|S| = \omega_X(G)$. Since $|S| = \delta_Y(G)$ every vertex in S is X-adjacent to at least one vertex in X-S. That is, X-S is a X-dominating set and hence a nonsplit X-dominating set. Therefore, $\gamma_{snsX}(G) \le p-\omega_X(G) \le p-\delta_Y(G)$. Already, $p-\omega_X(G) \le \gamma_{snsX}(G)$. Therefore, $\gamma_{snsX}(G) = p-\delta_Y(G)$.

Assume condition (ii). That is, $\omega_X(G) = \delta_Y(G)+1$ and every ω_X-set S contains a vertex not X-adjacent to any vertex of X-S. Let w in S be the vertex not X-adjacent to any vertex of X-S. Then $(X-S)\cup\{w\}$ is a nonsplit X-dominating set. Therefore, $\gamma_{snsX}(G) \le p-\omega_X(G)+1 \le p-\delta_Y-1+1 = p-\delta_Y(G)$. That is $\gamma_{snsX}(G) \le p-\delta_Y(G)$, since every ω_X set of cardinality δ_Y+1 contains a vertex not X-adjacent to any vertex of X-S. Therefore, $\gamma_{snsX}(G) \ge p-\delta_Y(G)$. Hence, $\gamma_{snsX}(G) = p-\delta_Y(G)$.

Conversely, let $\gamma_{snsX}(G) = p - \delta_Y(G)$. Then, $\omega_X(G) = \delta_Y(G)$ or $\omega_X(G) = \delta_Y(G) + 1$. Suppose there exists a ω_X - set with $|S| = \delta_Y(G) + 1$ such that every vertex in S is X-adjacent with some vertex in X-S. Then X-S is a strong nonsplit X-dominating set of G. Hence, $\gamma_{snsX}(G) \leq p - \delta_Y$, a contradiction. Hence, one of the given conditions is satisfied.

A. *Bipartite version of Strong nonsplit X-dominating set*

Given a graph, we can construct a variety of bipartite graph corresponding to the given graph. Here we define the bipartite graph VE(G)[1] constructed from G as follows: The graph VE(G)=(V,E,F) is a bipartite graph with the set of edges F defined as follows: x in V and e in E are adjacent if and only if x and e are incident with each other in G.

A subset D of V is a strong nonsplit dominating set if D is a dominating set and every vertex in V-D is adjacent. The minimum cardinality of a strong nonsplit dominating set of a graph G, denoted $\gamma_{sns}(G)$ is called the strong nonsplit domination number of a graph G.

Theorem 11: For any graph G, $\gamma_{snsX}(VE(G)) = \gamma_{sns}(G)$.

Proof: Let S be a γ_{snsX} $-$set of VE(G). The set S is X-dominating set and every vertex in X-S are X-adjacent. In the graph G, the set S is dominating set and the set V-S is a clique. Hence, S is a strong nonsplit dominating set in the graph G. Therefore, $\gamma_{sns}(G) \leq |S| = \gamma_{snsX}(VE(G))$.

Conversely, let us assume that D is γ_{sns} $-$set of G. The set D is dominating set and every vertex in X-D is adjacent. In the graph VE(G)= (X,Y,F), the set D of X is X-dominating and the set X-D is a X-clique. Therefore, D is a strong nonsplit X-dominating set in VE(G). Hence. $\gamma_{snsX}(VE(G)) \leq |S| = \gamma_{sns}(G)$.

IV. CONCLUSION

In this paper, minimal strong nonsplit X-dominating set is defined and is characterized. We have also calculated the bounds of the strong nonsplit X-domination number and has given the bipartite version of the strong nonsplit X-dominating set.

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