An efficient wavelet based approximation method for a few second order differential equations arising in science and engineering

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***Abstract*— A new wavelet based approximation method for solving the second order differential equations arising science and engineering is presented in this paper. Such differential equation is often applied to model phenomena in various fields of science and engineering. In this study, shifted second kind Chebyshev wavelet (CW) operational matrices of derivatives is introduced and applied for solving the second order differential equations with various initial conditions. The key idea for getting the numerical solutions for these equations is to convert the differential equations (linear or nonlinear) to a system of linear or nonlinear algebraic equations in the unknown expansion coefficients. Some illustrative examples are given to demonstrate the validity and applicability of the proposed method. The power of the manageable method is confirmed. Moreover the use of the shifted second kind Chebyshev wavelet method (CWM) is found to be simple, flexible, efficient, small computation costs and computationally attractive.**

Keywords: Second order differential equation, shifted second kind Chebyshev wavelet method, operational matrices, computationally attractive

I. INTRODUCTION

Wavelet analysis is a new branch of mathematics and widely applied in signal analysis, image processing and numerical analysis etc. The wavelet methods have proved to be very effective and efficient tool for solving problems of mathematical calculus. In recent years, wavelet transforms have found their way into many different fields in science, engineering and medicine. It possesses many useful properties, such as Compact support, orthogonality, dyadic, orthonormality and multi-resolution analysis (MRA). Recently, wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. After discretizing the differential equations in a conventional way like the finite difference approximation, wavelets can be used for algebraic manipulations in the system of equations obtained which lead to better condition number of the resulting system.

In the numerical analysis, wavelet based methods and hybrid methods become important tools because of the properties of localization. In wavelet based methods, there are two important ways of improving the approximation of the solutions: Increasing the order of the wavelet family and the increasing the resolution level of the wavelet. There is a growing interest in using various wavelets to study problems, of greater computational complexity. Among the wavelet transform families the Haar, Legendre and Chebyshev wavelets deserve much attention. The basic idea of Chebyshev wavelet method (CWM) is to convert the differential equations to a system of algebraic equations by the operational matrices of integral or derivative. Hariharan and co-workers [1-4] have introduced the Haar wavelet method for nonlinear reaction-diffusion equations arising in science and engineering. Padma et al. [5] have proposed the homotopy analysis method to water quality model in a uniform channel. The main goal is to show how wavelets and multi-resolution analysis can be applied for improving the method in terms of easy implementability and achieving the rapidity of its convergence. Chebyshev polynomials which are the eigen functions of a singular Sturm-Liouville problem have many advantages [6].

Beginning from 1991, wavelet technique has been applied to solve differential equations. Wavelets, as very well-localized functions, are considerably useful for solving differential equations and provide accurate solutions. Also, the wavelet technique allows the creation of very fast algorithms when compared with the algorithms ordinarily used [7].

In the present paper, the shifted second kind Chebyshev wavelet method (CWM) is used to compute the numerical solutions for the second order differential equations arising in science and engineering. These Chebyshev wavelets, which consist of Chebyshev polynomials, are given [8].

This work deals with the study of second order differential equation arising in science and engineering. Such differential equation is often used to model phenomena in scientific and technological problems.

Consider the second order differential equations of the form

$$\begin{cases} y''(t) = f(t, y, y') \\ y(\alpha) = \alpha \\ y'(\alpha) = \beta \end{cases} \tag{1}$$

where, $a \le t \le b, \quad a = t_0 < t_1 <,.... < t_{n-1} = b, \alpha, \beta \in R$

For example, consider the second order initial value problem

$$\left. \begin{aligned} & y'' + p(x)y' + q(x)y + N(y) = g(x) \\ & y'(\alpha) = \alpha \\ & y(\alpha) = \beta \end{aligned} \right\} \tag{2}$$

where α and β are constants. N(y) is a nonlinear term and g(x) is the source term.

Eq. (2) can be written in canonical form

$$Ly = -p(x)y - q(x)y - N(y) + g(x) \tag{3}$$

Where is the differential operator L is given by

$$L^{-1} = \frac{d^2}{dx^2}(.)$$

II. CHEBYSHEV WAVELETS PRELIMINARIES

It is well known that the second kind Chebyshev polynomials are defined on [-1,1] by

$$U_n(x) = \frac{\sin\,(n+1)\theta}{\sin\,\theta}, \ x = \cos\,\theta. \tag{4}$$

These polynomials are orthogonal on [-1,1].

That is,

$$\int_{-1}^{1} \sqrt{1-x^2}\, U_m(x) U_n(x)\, dx = \begin{cases} 0, m \ne n \\ \dfrac{\pi}{2} \qquad m = n \end{cases} \tag{5}$$

The following properties of second kind Chebyshev polynomials [9] are of fundamental importance in the sequel. They are eigen functions of the following singular Sturm-Liouville equation.

$$(1-x^2)D^2\phi_k(x) - 3xD\phi_k(x) + k(k+2)\phi_k(x) = 0, \tag{6}$$

Where $D \equiv \dfrac{d}{dx}$ and may be generated by using the recurrence relation

$$U_{k+1}(x) = 2xU_k(x) - U_{k-1}(x), k = 1,2,3,... \tag{7}$$

Starting from $U_0(x) = 1$ and $U_1(x) = 2x$, or from Rodrigues formula

$$U_n(x) = \frac{(-2)^n (n+1)!}{(2n+1)!\sqrt{(1-x^2)}} D^n[(1-x^2)^{n+\frac{1}{2}}] \tag{8}$$

The following theorem is needed hereafter.

Theorem 2.1: The first derivative of second kind Chebyshev polynomials is given by

$$DU_n(x) = 2 \sum_{\substack{k=0 \\ (k+n)odd}}^{n-1} (k+1)U_k(x). \tag{9}$$

A. Shifted second kind Chebyshev polynomials

The shifted second kind Chebyshev polynomials are defined on [0,1] by $U_n^*(x) = U_n(2x-1)$. All results of second kind Chebyshev polynomials can be easily transformed to give the corresponding results for their shifted forms. The orthogonality relation with respect to the weight function $\sqrt{x-x^2}$ is given by

$$\int_0^1 \sqrt{x-x^2}\,U_n^*(x)U_m^*(x)dx = \begin{cases} 0, & m \neq n \\ \frac{\pi}{8}, & m = n. \end{cases}$$

The first derivative $U_n^*(x)$ is given by the following corollary.

Corollary 1: The first derivative of the shifted second kind Chebyshev polynomial is given by

$$DU_n^*(x) = 4 \sum_{\substack{k=0 \\ (k+n) odd}} (k+1)U_k^*(x) \tag{10}$$

B. Shifted second kind Chebyshev operational matrix of derivatives

Wavelets constitute of a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter a and the translation parameter b varies continuously, then we have the following family of continous wavelets;

$$\psi_{a,b}(t) = |a|^{-1/2}\,\psi\left(\frac{t-b}{a}\right) \quad a,b \in R, \quad a \neq 0 \tag{11}$$

Let $\psi_{n,m}(t) = \psi(k,n,m,t)$ is the second kind Chebyshev wavelets. Here k,n can assume any positive integers, m is the order of second kind Chebyshev wavelet and t is the normalized time.

It is defind on the interval [0,1] by

$$\psi_{n,m}(t) = \begin{cases} \frac{2^{\frac{k+3}{2}}}{\sqrt{\pi}} U_m^*(2^k t - n), & t \in \left[\frac{n}{2^k}, \frac{n+1}{2^k}\right], \\ 0 & otherwise \end{cases} \tag{12}$$

m= 0,1,…..M, n= 0,1,…2^k-1. A function f(t) defined over [0,1] may be expanded in terms second kind Chebyshev wavelets as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t). \tag{13}$$

Where

$$c_{nm} = \left(f(t), \psi_{nm}(t)\right)_w = \int_0^1 \sqrt{t-t^2}\, f(t)\psi_{nm}(t)dt. \tag{14}$$

If the infinite series is truncated, then it can be written as

$$f(t) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} c_{nm}\psi_{nm}(t). = C^T\psi(t) \tag{15}$$

Where C and ψ(t) are $2^k(M+1)$ x 1 defined by

$$\left.\begin{array}{l} C=[c_{0,0},c_{0,1},,...,c_{0,M},....,c_{2^k-1,M},...,c_{2^k-1,1},....,c_{2^k-1,M}]^T \\ \psi(t)=[\psi_{0,0},\psi_{0,1},,...,\psi_{0,M},....\psi_{2^k-1,M},...,\psi_{2^k-1,1},....,\psi_{2^k-1,M}]^T \end{array}\right\} \qquad (16)$$

A shifted second kind Chebyshev wavelets operational matrix of the first derivative is stated and proved in the following theorem.

Theorem 2.2:

Let ψ(t) be the second kind chebyshev wavelets vector defined in Eq.(16). Then the first derivative of the vector ψ(t) can be expressed as

$$\frac{d\psi(t)}{dt}=D\,\psi(\mathrm{t}) \qquad (17)$$

where D is $2^k(M+1)$ square matrix of derivatives and is defined by

$$D=\begin{bmatrix} F & O & . & . & . & O \\ O & F & . & . & . & O \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ O & O & . & . & . & F \end{bmatrix} \qquad (18)$$

in which F is an (M+1) square matrix and its (r,s)th element is defined by

$$F_{r,s}=\begin{cases} 2^{k+2}s & r\geq 2,\quad r>s \quad and \quad (r+s) \quad odd. \\ 0, & otherwise \end{cases}$$

Corollary 2.1. The operational matrix for the n^{th} derivative can be obtained from

$$\frac{d^n\psi(t)}{dt^n}=D^n\psi(t),\quad n=1,2,...where\ D^n\ is\ the\ n^{th}\ power\ of\ D. \qquad (19)$$

C. Second- order two-point boundary value problems

Consider the linear second-order differential equation

$$y''(x)+f_1(x)y'(x)+f_2(x)y(x)=g(x),\qquad x\in[0,1], \qquad (20)$$

Subject to the initial conditions

$$y(0)=\alpha,\qquad y'(0)=\beta \qquad (21)$$

or the boundary conditions

$$y(0)=\alpha,\qquad y(1)=\beta \qquad (22)$$

or the most general mixed boundary conditions

$$\alpha_1 y(0)+\alpha_2 y'(0)=\alpha,\qquad b_1 y(1)+b_2 y'(1)=\beta. \qquad (23)$$

If we approximate y(x), $f_1(x)$, $f_2(x)$ and g(x) in terms of the second kind Chebyshev wavelet basis, then one can write

$$y(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} c_{nm}\psi_{nm}(x) = C^T\psi(x). \qquad f_1(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} f_{nm}\psi_{nm}(x) = F_1^{\,T}\psi(x)$$

$$f_2(x) \approx \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} f_{nm}\psi_{nm}(x) = F_2^{\,T}\psi(x) \qquad g(x) = \sum_{n=0}^{2^k-1}\sum_{m=0}^{M} g_{nm}\psi_{nm}(x) = G^T\psi(x) \tag{24}$$

approximate

where C^T, F_1^T, F_2^T and G^T are defined similarly as in (7).Relations ( ) and ( ), enable one to $y'(x)$ and $y''(x)$ as

$$y'(x) \approx C^T D\psi(x), \qquad y''(x) = C^T D^2\psi(x) \tag{25}$$

Now substitution of relations (24) and (25) into Eq. (20), enable us to define the residual, R(x), of this equation as

$$R(x) = C^T D^2\psi(x) + F_1^{\,T}\psi(x)(\psi(x))^T D^T C + F_2^{\,T}\psi(x)(\psi(x)^T C - G^T\psi(x). \tag{26}$$

and application of the tau method (see, Ref.[10]), yields the following (2^k(M+1) -2) linear equations in the unknown expansion coefficients, c_{nm}, namely

$$\int_0^t \sqrt{x-x^2}\,\psi_j(x)R(x)dx = 0, \qquad j = 1,2,...2^k(M+1)-2. \tag{27}$$

Moreover, the initial conditions Eq.(21), the boundary conditions Eq.(22), and the mixed boundary conditions Eq.(23) lead respectively, to the following equations

$$C^T\psi(0) = \alpha, \qquad C^T D\psi(0) = \beta, \tag{28}$$

$$C^T\psi(0) = \alpha \qquad C^T\psi(1) = \beta \tag{29}$$

and

$$\left.\begin{aligned} a_1 C^T\psi(0) + a_2 D\psi(0) = \alpha, \\ b_1 C^T\psi(1) + b_2 C^T D\psi(1) = \beta \end{aligned}\right\} \tag{30}$$

Thus Eq. (27) with the two equations Eq.(28) or (29) or (30) generate 2^k(M+1) a set of linear equations which can be solved for the unknown components of the vector C, and hence an approximate spectral wavelets solution to y(x) given in Eq.(24) can be obtained.

D. Nonlinear second-order two-point boundary value problems

Consider the nonlinear differential equation

$$y''(x) = F(x, g(x), y'(x)), \tag{31}$$

Subject to the initial conditions

$$y(0) = \alpha, \qquad y'(0) = \beta$$

or the boundary conditions

$$y(0) = \alpha, \qquad y(1) = \beta$$

or the most general mixed boundary conditions

$$\alpha_1 y(0) + \alpha_2 y'(0) = \alpha, \qquad b_1 y(1) + b_2 y'(1) = \beta.$$

If we follow the same procedure

If we follow the same procedure of Section (2.4) approximate y(x) as in (24) and make use of (18) and (19), then we obtain

$$C^T D^2 \psi(t) = F(x, y(x), C^T D \psi(x)) \tag{32}$$

To find an approximate solution to y(x), we compute (32) at the first $2^k(M+1) - 2$ roots of

$U^*_{2^k(M+1)}(x)$.

These equations with the two Eqs. (28) or (29) or (30) generate $2^k(M+1)$ non linear equations in the expansion coefficients C_{nm} which can be solved with the aid of Newton's iterative method.

III. NUMERICAL EXAMPLES

Example 1: Consider the Painleve Equation of the form [11]

$$\frac{d^2u}{dx^2} = 6u^2(x) + x \qquad x \in [0,1) \tag{33}$$

With the initial condition given by

$$u(0) = 1; u'(0) = 0$$

We solve the Eq. (33) using the algorithm described in section (2.5) for the case corresponds to M=2, k=0 to obtain an approximate solution of u(x). First, if we make use of (18) and (19), then the two operational matrices D and D^2 are given respectively by

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 4 & 0 & 0 \\ 0 & 8 & 0 \end{bmatrix} \qquad D^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 32 & 0 & 0 \end{bmatrix}$$

Moreover $\psi(x)$ can be evaluated to give

$$\psi(x) = \sqrt{\frac{2}{\pi}} \begin{pmatrix} 2 \\ 8x - 4 \\ 32x^2 - 32x + 6 \end{pmatrix}$$

If we set

$$C = (c_{0,0}, c_{0,1}, c_{0,2})^T = \sqrt{\frac{\pi}{2}}(c_0, c_1, c_2)^T$$

Then Eq. (33), takes the form

$$C^T D^2 \psi(x) - 6(C^T \psi(x))^2 - G^T \psi(x) = 0$$

Which is equivalent to

$$64\,C_2 - 6(2C_0 + (8x-4)\,C_1 + (32\,x^2 - 32\,x + 6)C_2)^2 - \sqrt{2\pi}(\frac{2}{8} + \frac{1}{16}(8x-4)) = 0$$

We only need to satisfy this equation at the first root of $U_3^*(x)$, i.e., at $x = \frac{2-\sqrt{2}}{4}$, to get

$$64\,C_2 - 6(2C_0 - 2\sqrt{2}C_1 + 2\,C_2)^2 - \frac{\sqrt{2\pi}}{4}(1 - \frac{\sqrt{2}}{2}) \tag{34}$$

Furthermore, the use of initial conditions in Eq.(33) lead to the two equations

$$2\,C_0 - 4\,C_1 + 6C_2 = 0 \tag{35}$$

$$C_1 - 4C_2 = 0 \tag{36}$$

The solution of the nonlinear system of Eqs.(35) and (36) gives

$C_0 = 0.67835, \qquad C_1 = 0.14268 \quad and \quad C_2 = 0.356$

Consequently

$$y(x) = \begin{pmatrix}0.67835 & 0.14268 & 0.03567\end{pmatrix} \begin{pmatrix} 2 \\ 8x-4 \\ 32x^2 - 32x + 6 \end{pmatrix} \tag{37}$$

$= 1.1414\,x^2 + 1$

When $C_2 = 0.03567$, $\quad C_1 = 0.38196$, $\quad C_0 = 0.97745$

$$y(x) = \begin{pmatrix}0.97745 & 0.38196 & 0.09549\end{pmatrix} \begin{pmatrix} 2 \\ 8x-4 \\ 32x^2 - 32x + 6 \end{pmatrix}$$

$$= \quad 3.05568x^2 + 1$$

which is the exact solution.

Example 2: We consider the equation

$$\begin{aligned} &u'' + u(x) = x \qquad 0 \le x \le 1 \\ &u(0) = u'(0) = 1 \end{aligned} \tag{38}$$

With the exact solution $u(x) = x + \cos x$

Then Eq. (38) takes the form

$C^T D^2 \psi(x) + C^T \psi(x) - G^T \psi(x) = 0$

which is equivalent to

$64\,C_2 + 2\,C_0 + (8x-4)C_1 + (32\,x^2 - 32\,x + 6)C_2 + x = 0$

We only need to satisfy this equation at the first root of $U_3^*(x)$ i.e., at $x = \dfrac{2-\sqrt{2}}{4}$, to get

$$2C_0 - 2\sqrt{2}C_1 + 66\,C_2 = \frac{2-\sqrt{2}}{4} \tag{39}$$

Furthermore, the set of initial conditions in (38) lead to the equations

$$2\,C_0 - 4\,C_1 + 6C_2 = 1 \tag{40}$$

$$8C_1 - 32C_2 = 1 \tag{41}$$

The solution of the nonlinear system of equations (40) and (41) gives

$C_0 = 0.6727, \qquad C_1 = 0.06316, \qquad C_2 = -0.01545$

Consequently

$$u(x) = \begin{pmatrix}0.6727 & 0.06316 & -0.01545\end{pmatrix} \begin{pmatrix} 2 \\ 8x-4 \\ 32x^2 - 32x + 6 \end{pmatrix}$$

$$= -0.4947x^2 + x + 1.0001 \tag{42}$$

Example 3: Consider the nonlinear oscillator problem [12]

$$u''(x) - u(x) + u^2(x) + (u'(x))^2 - 1 = 0, \qquad 0 \le x \le 1$$
$$u(0) = 2 \tag{43}$$
$$u'(0) = 0$$

Theoretical solution u(x) = 1 + cos x

We solve the Eq. (44) using the algorithm described in section (2.5)

Equation takes the form

$$C^T D^2 \psi(x) - C^T \psi(x) + (C^T \psi(x))^2 + (C^T D \psi(x))^2 - 1 = 0$$

which is equivalent to

$$64\, C_2 - (2\, C_0 + (8x-4)\, C_1 + (32\, x^2 - 32\, x + 6)\, C_2) + (2\, C_0 + (8x-4)\, C_1 + (32\, x^2 - 32\, x + 6)\, C_2)^2 + (8\, C_1 + 8\,(8x-4)\, C_2)^2 - 1 = 0$$

We only need to satisfy this equation at the first root of $U_3^*(x)$ i.e., at $x = \frac{2-\sqrt{2}}{4}$, to get

$$64\, C_2 - (\,2\, C_0 - 2\sqrt{2}\, C_1 + 2\, C_2) + (2\, C_0 - 2\sqrt{2}\, C_1 + 2\, C_2)^2 + 64\,(C_1 - 2\sqrt{2}\, C_2)^2 - 1 = 0 \tag{44}$$

Furthermore, the set of initial conditions in (44) lead to the equations

$$2\, C_0 - 4\, C_1 + 6\, C_2 = 2 \tag{45}$$

$$8C_1 - 32C_2 = 0$$

Consequently

$$u(x) = \begin{pmatrix} 1.0755 & 0.0604 & 0.0151 \end{pmatrix} \begin{pmatrix} 2 \\ 8x-4 \\ 32x^2 - 32x + 6 \end{pmatrix}$$
$$= 2 - 0.4832x^2 \tag{46}$$

IV. Conclusion

In this paper we have applied new methods called the shifted second kind Chebyshev wavelet method and have used it to derive the exact and approximate analytical solutions of second order differential equations arising in science and engineering. We have established that this method is capable of reducing the volume of the computational work as compared to the other classical methods. The key idea of this approach is that it reduces the differential equations to a system of algebraic equations. Thus, we conclude that the proposed method can be considered as nice refinement in existing numerical techniques and might have wide applications. Finally, three examples were presented and their numerical results agreed well with the exact solution.

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