A New Polynomial Method for Solving Fredholm –Volterra Integral Equations

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***Abstract*— A new polynomial method to solve Volterra–Fredholm Integral equations is presented in this work. The method is based upon Shifted Legendre Polynomials. The properties of Shifted Legendre Polynomials and together with Gaussian integration formula are presented and are utilized to reduce the computation of Volterra–Fredholm Integral equations to a system of algebraic equations. Some numerical examples are selected to illustrate the proposed method also the theoretical analysis of shifted Legendre polynomial method such as convergence and error analysis has been discussed. The results demonstrate reliability and efficiency of the proposed method.**

Keyword- Shifted Legendre Polynomials; Nonlinear Integro-Differential equations; Volterra–Fredholm Integral equations; Approximate Solution; Convergence Analysis.

I. INTRODUCTION

Mathematical modeling of physical phenomena namely fluid dynamics, biological models, chemical kinetics and other disciplines lead to linear / nonlinear integro-differential equations. The various types of linear / nonlinear integro-differential equations particularly Fredholm, Volterra, Volterra–Hammerstein, Fredholm-Volterra, impulse integro-differential equations and singular integro-differential equations with their solution methods are reported in [1-36].

These methods are divided into two category namely analytical methods and numerical methods. The Fourier function [2], Adomian decomposition method [13], Homotopy perturbation method [13,14,18-20,23,24,27] are some of the analytical methods used to solve the integral equations . The theoretical analysis such as convergence and error analysis has been discussed in detailed manner. The numerical methods for solving nonlinear integro-differential are Galerkin technique [8,16,23], piecewise interpolation [9], collocation-type method or iterative method [5-7, 32-35], wavelets [3,4,11,12,29] and other methods [10, 31,36] that provide error analysis for particular problem type.

Recently, Legendre polynomial based methods [37-43]are used to obtain the fast solutions problems of science and engineering. The main characteristic behind in this technique is that it reduces these problems to those of solving a system of linear /nonlinear algebraic equations thus greatly simplifying the problem. In this Legendre polynomial method, a truncated orthogonal series is used for numerical integration of differential equations, with a goal of obtaining efficient computational solutions. Shifted Legendre polynomial method SLPM [44] is the next version of Legendre polynomial with shifting property to shift the interval from [-1, 1] into [0, 1].

The various applications of shifted Legendre polynomials had been studied by many researchers and several papers [44-48] have appeared in the literature, concerned with operational matrix based techniques. Theoretical analyses such as error analysis and convergence have also been discussed to demonstrate its potential use.

In this paper , we intend to extend the application of SLPM to find the approximate solution of a nonlinear Fredholm–Volterra integral equation[38]

$$y(x) = f(x) + \lambda_1 \int_0^x K_1(x,t)\,F(y(t))\,dt \quad + \lambda_2 \int_0^1 K_2(x,t)\,G(y(t))\,dt,\ \ 0 \le x,t \le 1, \tag{1}$$

where f (x), the kernels $K_1(x, t)$ and $K_2(x, t)$ are assumed to be in $L^2(R)$ on the interval $0 \le x, t \le 1$, which is the general form of [49,50].

The method consists of expanding the solution by shifted Legendre polynomial with unknown coefficients. The properties of shifted Legendre polynomial together with the Gaussian integration formula [30] are then utilized to evaluate the unknown coefficients and we find an approximate solution to Eq. (1).

This paper is organized as follows. In Section 2, Properties of Shifted Legendre polynomials are discussed. Section 3 explains the solution of the nonlinear Volterra–Fredholm integral equation and Section 4 and 5 provide the convergence and error analysis of SLPM. In Section 6, we demonstrate the accuracy of the proposed scheme by considering numerical examples. The salient features of SLPM and concluding remarks are discussed in Section 7 and 8 respectively.

II. SHIFTED LEGENDRE POLYNOMIALS AND ITS PROPERTIES

In this section, we discuss Legendre polynomials and its function approximation[37-43].

A. *Legendre polynomials*

The Legendre polynomials are defined on the interval [-1, 1] and can be determined with the aid of the following recurrence formulae:

$L_{i+1}(z)=\frac{2i+1}{i+1}zL_i(z)-\frac{i}{i+1}L_{i-1}(z), \qquad i=1,2,...,$ where $L_0(z)=1$ and $L_1(z)=z$.

B. Shifted Legendre polynomials

In order to use these polynomials on the interval $x\in[0,1]$, recently Saadatmandi [44] utilized the so-called shifted Legendre polynomials by introducing the change of variable $z=2x-1$.

Let the shifted Legendre polynomials $L_i(2x-1)$ be denoted by $P_i(x)$. Then $P_i(x)$ can be obtained as follows:

$$P_{i+1}(x)=\frac{(2i+1)(2x-1)}{(i+1)}P_i(x)-\frac{i}{i+1}P_{i-1}(x), \qquad i=1,2,..., \tag{2}$$

where $P_0(x)=1$ and $P_1(x)=2x-1$. The analytic form of the shifted Legendre polynomial $P_i(x)$ of degree i given by

$$P_i(x)=\sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2}. \tag{3}$$

Note that $P_i(0)=(-1)^i$ and $P_i(1)=1$. The orthogonality condition is

$$\int_0^1 P_i(x)P_j(x)dx=\begin{cases}\dfrac{1}{2i+1} & for\ i=j,\\ 0 & for\ i\neq j.\end{cases} \tag{4}$$

A function y(x), square integrable in [0,1], may be expressed in terms of shifted Legendre polynomials as

$$y(x)=\sum_{j=0}^{\infty}c_jP_j(x),$$

where the coefficients c_j are given by $c_j=(2j+1)\int_0^1 y(x)P_j(x)dx, \quad j=1,2,....$

C. Function approximation

In practice, only the first (m+1) terms of shifted Legendre polynomials are considered. Then we have

$$y(x)=\sum_{j=0}^{m}c_jP_j(x)=C^T\Phi(x), \tag{5}$$

where the shifted Legendre coefficient vector C and the shifted Legendre vector $\Phi(x)$ are given by

$$C^T=[c_0,...,c_m],$$
$$\Phi(x)=[P_0(x),P_1(x),...,p_m(x)]^T. \tag{6}$$

III. SOLUTION OF THE NONLINEAR VOLTERRA–FREDHOLM INTEGRAL EQUATIONS

Consider the nonlinear Volterra–Fredholm integral equations given in Eq. (1). In order to use Legendre polynomials, we first approximate y(x) as

$$y(x)=C^T\Phi(x), \tag{7}$$

From Eqs. (1) and (7) we have

$$C^T\Phi(x) = f(x) + \lambda_1 \int_0^x K_1(x,t) F(C^T\Phi(t))dt \ + \lambda_2 \int_0^1 K_2(x,t)\, G(C^T\Phi(t))dt \tag{8}$$

We now collocate Eq.(8) at m+1 points x_i as

$$C^T\Phi(x_i) = f(x_i) + \lambda_1 \int_0^{x_i} K_1(x_i,t) F(C^T\Phi(t))dt \ + \lambda_2 \int_0^1 K_2(x_i,t)\, G(C^T\Phi(t))dt \tag{9}$$

Suitable collocation points are zeros of Chebyshev polynomials [30]

$$x_i = \cos\left(\frac{(2i+1)\pi}{m+1}\right), \qquad i = 1 \ to\, m+1\,.$$

In order to use the Gaussian integration formula for Eq. (9), we transfer the t-intervals $[0,x_i]$ and [0, 1] into τ_1 and τ_2 intervals [−1, 1] by means of the transformations

$$\tau_1 = \frac{2}{x_i}t - 1, \qquad \tau_2 = 2t - 1,$$

Let

$$H_1(x_i,t) = K_1(x_i,t)\,F(C^T\Phi(t)),$$

$$H_2(x_i,t) = K_2(x_i,t)\,G(C^T\Phi(t))$$

Eq. (9) may then be restated as

$$C^T\Phi(x_i) = f(x_i) + \lambda_1 \frac{x_i}{2}\int_{-1}^{1} H_1\left(x_i, \frac{x_i}{2}(\tau_1+1)\right)d\tau_1 + \frac{\lambda_2}{2}\int_{-1}^{1} H_2\left(x_i, \frac{1}{2}(\tau_2+1)\right)d\tau_2$$ By using the Gaussian integration formula we get

$$C^T\Phi(x_i) \approx f(x_i) + \lambda_1 \frac{x_i}{2}\sum_{j=1}^{s_1}\omega_{1j} H_1\left(x_i, \frac{x_i}{2}(\tau_{1j}+1)\right) + \frac{\lambda_2}{2}\sum_{j=1}^{s_2}\omega_{2j} H_2\left(x_i, \frac{1}{2}(\tau_{2j}+1)\right) \quad \text{i= 1 to M+1,} \tag{10}$$

where τ_{1j} and τ_{2j} are s_1 and s_2 zeros of Legendre polynomials P_{s_1+1} and P_{s_2+2} respectively , and w_{1j} , w_{2j} are the corresponding weights given in [12].

The idea behind the above approximation is the exactness of the Gaussian integration formula for polynomials of degree not exceeding 2_{s_1+1} and 2_{s_2+2} . Eq. (10) gives M+1 nonlinear algebraic equation which can be solved for the elements of C^T in Eq. (7) using Newton's iterative method.

IV. CONVERGENCE ANALYSIS

Theorem 4.1: Convergence theorem

The series solution Eq.(5) of Eq. (1) using SLPM converges towards y(x).

Proof:

Let $L^2(R)$ be the Hilbert space and $P_i(x) = \sum_{k=0}^{i}(-1)^{i+k}\frac{(i+k)!}{(i-k)!}\frac{x^k}{(k!)^2}$

Let $y(x) = \sum_{j=0}^{m} c_j P_j(x)$

where $c_j = (2j+1)\int_0^1 y(x)P_j(x)\,dx \ \ c_j = (2j+1)\langle y(x), P_j(x)\rangle$

where $\langle .\ ,\ .\ \rangle$ represents an inner product.

$$y(x) = (2j+1)\sum_{i=1}^{n}\langle y(x), P_j(x)\rangle\ P_j(x)$$

where j=1,2,3, …

Let $\alpha_j = \langle y(x), P_j(x)\rangle$

Define the sequence of partial sums $\{S_n\}$ of $\left(\alpha_j P_j(x)\right)$; let S_n and S_m be arbitrary partial sums with $n \geq m$. We are going to prove that $\{S_n\}$ is a Cauchy sequence in Hilbert space.

Let $S_n = \sum_{j=1}^{n} \alpha_j P_j(x)$ $\langle y(x), S_n \rangle = (2j+1)\left\langle y(x), \sum_{j-1}^{n} \alpha_j P_j(x) \right\rangle$

$$= (2j+1)\sum_{j=1}^{n} \overline{\alpha_j} \langle y(x), P_j(x) \rangle$$

$$= (2j+1)\sum_{j-1}^{n} \overline{\alpha_j}\, \alpha_j$$

$$= (2j+1)\sum_{j-1}^{n} |\alpha_j|^2$$

We will claim that $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} \frac{|\alpha_j|^2}{2j+1}$ for n > m.

(ie) $$\|S_n - S_m\|^2 = \left\| \sum_{j=m+1}^{n} \alpha_j P_j(x) \right\|^2$$

$$= \left\langle \sum_{i=m+1}^{n} \alpha_i P_i(x), \sum_{j=m+1}^{n} \alpha_j P_j(x) \right\rangle$$

$$= \sum_{i=m+1}^{n} \sum_{j=m+1}^{n} \alpha_i \overline{\alpha_j} \langle P_i(x), P_j(x) \rangle$$

$$= \sum_{j=m+1}^{n} \alpha_j \overline{\alpha_j} \left(\frac{1}{2j+1} \right)$$

$$= \sum_{j=m+1}^{n} \frac{|\alpha_j|^2}{2j+1}$$

i.e $\|S_n - S_m\|^2 = \sum_{j=m+1}^{n} \frac{|\alpha_j|^2}{2j+1}$ for n > m.

From Bessel's inequality, we have $\sum_{j=1}^{\infty} |\alpha_j|^2$ is convergent and hence

$\|S_n - S_m\|^2 \to 0 \quad as\ m, n \to \infty.$

i.e $\|S_n - S_m\| \to 0$ and $\{S_n\}$ is a Cauchy sequence and it converges to say 's'.

We assert that y(x) = s

Infact, $\langle s - y(x), P_j(x) \rangle = \langle s, P_j(x) \rangle - \langle y(x), P_j(x) \rangle$

$$= \left\langle \underset{n\to\infty}{Lt}\, S_n, P_j(x) \right\rangle - \alpha_j$$

$$= \underset{n\to\infty}{Lt} \langle S_n, P_j(x) \rangle - \alpha_j$$

$$= \alpha_j - \alpha_j$$

$\Rightarrow \langle s - y(x), P_j(x) \rangle = 0$

Hence $y(x)=s$ and $\sum_{j=0}^{m} C_j P_j(x)$ converges to y(x) and this completes the proof. As the convergence has been proved, consistency and stability are ensured automatically.

V. ERROR ANALYSIS

In this part, an error estimation for the approximate solution of Eq.(1) is discussed. Let us consider $e_n(x)=y(x)-\overline{y}(x)$ as the error function of the approximate solution $\overline{y}(x)$ for $y(x)$, where $y(x)$ is the exact solution of Eq.(1).

$$y(x)=\sum_{j=0}^{\infty} c_j P_j(x) \tag{11}$$

and $\overline{y}(x)=\sum_{j=0}^{m} c_j P_j(x) = C^T \Phi(x)$

$\overline{y}(x)=C^T\Phi(x)+H_n(x)$ where $\mathrm{H}_n(x)$ is the perturbation term.

$$H_n(x)=\overline{y}(x)-C^T\Phi(x). \tag{12}$$

We proceed to find an approximation $\overline{e_n}(x)$ to the error function $e_n(x)$ in the same way as we did before for the solution of the problem. Subtracting Eq. (12) from Eq. (11), the error function $e_n(t)$ satisfies the problem.

$$e_n(x)-C^T\Phi(x)=-\mathrm{H}_n(x) \tag{13}$$

It should be noted that in order to construct the approximate $\overline{e_n}(x)$ to $e_n(x)$, only Eq. (13) needs to be recalculated in the same way as we did before for the solution of Eq.(11).

VI. IIUSTRATIVE EXAMPLES

In this section, we examine the performance of SLPM with some examples.

Example 1

Consider the nonlinear Volterra–Fredholm integral equation given in [38] by

$$y(x)=-\frac{1}{30}x^6+\frac{1}{3}x^4-x^2+\frac{5}{3}x-\frac{5}{4},$$
$$+\int_0^x (x-t)[y(t)]^2\,dt+\int_0^1 (x+t)[y(t)]dt$$
$$0\le x,\, t\le 1. \tag{14}$$

We apply the method presented in this paper and solve Eq.(14) with m=3 we get the following algebraic equations

$(1/2)c_0-(7/6)c_1+c_2-c_3=-(5/4)$

$-c_0+2c_1-6c_2+12c_3=(5/3)$

$(1/2)c_0^2+(1/2)c_1^2+(1/2)c_2^2+(1/2)c_3^2-c_0c_1+c_0c_2-c_0c_3-c_1c_2+c_1c_3-c_2c_3-6c_2+30c_3=1$

$(-2/3)c_1^2-2c_2^2-4c_3^2+(2/3)c_0c_1-2c_0c_2+4c_0c_3+(8/3)c_1c_2-(14/3)c_1c_3+6c_2c_3-20c_3=0$

From these equations we find $c_0=-1.6667$, $c_1=0.5$, $c_2=0.1667$, $c_3=0$ and

$y(x)=c_0\,p_0+c_1\,p_1+c_2\,p_2+c_3\,p_3=x^2-2$, which is the exact solution.

Example 2

Consider the nonlinear Volterra integral equation given in [38] by

$$y(x)=\exp(x)-\frac{1}{3}\exp(3x)+\frac{1}{3}+\int_0^x [y(t)]^3\,dt, \tag{15}$$

We apply the method presented in this paper and solve Eq.(15) with m=2 we get the following algebraic equations;

$c_0-c_1+c_2=1$

$c_0^3-c_1^3+c_2^3-3c_0^2c_1+3c_0c_1^2+3c_1^2c_2-$
$3c_1c_2^2+3c_0^2c_2+3c_0c_2^2-6c_0c_1c_2-2c_1+6c_2=0$

$3c_1^3-9c_2^3+3c_0^2c_1-6c_0c_1^2-15c_1^2c_2+21c_1c_2^2-9c_0^2c_2-$

$18c_0c_2^2+24c_0c_1c_2-6c_2=1$

From these equations we find $c_0 = 1.6667$, $c_1 = 0.75$, $c_2 = 0.0833$ and y(x) = $1+x+\frac{x^2}{2}$

The closed solution of y(x) = e^x which is the exact solution for larger values of m. TABLE 1 shows the error for example 2 for different values of m with exact solution.

TABLE 1
The errors for Example 2 at m=4,10,15

m	*X*	*Exact*	*SLPM*	*Error*
4	*0*	*1*	*1*	*0*
	0.2	*1.221403*	*1.2213*	*0.0001027*
	0.4	1.491825	1.4907	0.0011246
	0.6	1.822119	1.816	0.0061188
	0.8	2.225541	2.2053	0.0202409
	1	2.718282	2.6667	0.0512818
10	0	1	1	0
	0.2	1.221403	1.2214	0.0000027
	0.4	1.491825	1.4918	0.0000246
	0.6	1.822119	1.8221	0.0000188
	0.8	2.225541	2.2255	0.0000409
	1	2.718282	2.7183	-0.0000182
15	0	1	1	0
	0.2	1.221403	1.2214	0.0000027
	0.4	1.491825	1.4918	0.0000246
	0.6	1.822119	1.8221	0.0000188
	0.8	2.225541	2.2255	0.0000409
	1	2.718282	2.7183	-0.0000182

Example 3

Consider the Integro-differential equation[51]

$$y'(x)=\int_0^1 x\,s\,y(s)\,ds+y(x)-\frac{4}{3}x+1;\quad y(0)=0 \tag{16}$$

We apply the method presented in this paper and solve Eq.(16) with m=3

(ie) $y(x)=\sum_{j=0}^{3} c_j P_j(x)$ we get the following algebraic equations

$c_0 - 3c_1 + 7c_2 - 13c_3 = -1$

$(1/2)c_0 + (13/6)c_1 - 18c_2 + 72c_3 = (4/3)$

$6c_2 - 90c_3 = 0$

and from the initial condition y(0)=0, we have $c_0 - c_1 + c_2 - c_3 = 0$

From these equations we find $c_0 = 0.5$, $c_1 = 0.5$, $c_2 = 0$, $c_3 = 0$.

Using Eq. (6) we get y(x)=$c_0 p_0 + c_1 p_1 + c_2 p_2 + c_3 p_3$ = x, which is the exact solution.

Example 4

Consider the Integro-differential equation [51]

$$y'(x) = \int_0^1 x\, y(s) ds + y(x) + e^x - x \qquad y(0)=0 \qquad (17)$$

We apply the method presented in this paper and solve Eq.(17) with m=3 we get
the following algebraic equations

$c_0 - c_1 + c_2 - c_3 = 0$

$c_0 - 3c_1 + 7c_2 - 13c_3 = -1$

$c_0 + 2c_1 - 18c_2 + 72c_3 = 0$

$6c_2 - 90c_3 = -(1/2)$

From these equations we find $c_0 = 0.9474$, $c_1 = 1.2079$, $c_2 = 0.2851$, $c_3 = 0.0246$.

The closed solution of y(x) = xe^x which is the exact solution for larger values of m. . TABLE 2 shows the error for example 2 for different values of m with exact solution.

TABLE 2
The errors for Example 4 at m=4,10,15

m	*X*	*Exact*	*SLPM*	*Error*
4	0	0	0	0
	0.2	0.244281	0.244	2.81E-04
	0.4	0.59673	0.592	4.73E-04
	0.6	1.093271	1.068	0.02527
	0.8	1.780433	1.696	0.08443
	1	2.718282	2.5	0.21828
10	0	0	0	0
	0.2	0.244281	0.2443	-2E-05
	0.4	0.59673	0.5967	3.00E-10
	0.6	1.093271	1.0933	1.77E-08
	0.8	1.780433	1.7804	3.21E-07
	1	2.718282	2.7183	-1.8E-05
15	0	0	0	0
	0.2	0.244281	0.2443	-2E-05
	0.4	0.59673	0.5967	2.98E-05
	0.6	1.093271	1.0933	-2.9E-05
	0.8	1.780433	1.7804	3.27E-05
	1	2.718282	2.7183	-1.2E-05

Example 5

Consider the Integro-differential equation[51]

$$y(x) = x + \int_0^x (t - x)\, y(t) dt \qquad (18)$$

We apply the method presented in this paper and solve Eq.(18) with m=3 we get
the following algebraic equations

$c_0 - c_1 + c_2 - c_3 = 0$

$2c_1 - 6c_2 + 12c_3 = 1$

$(-1/2)c_0 + (1/2)c_1 - (13/2)c_2 + (61/2)c_3 = 0$

$(1/3)c_1 - c_2 + 22c_3 = 0$

From these equations we find $c_0 = 0.4583$, $c_1 = 0.4250$, $c_2 = -0.0417$, $c_3 = -0.0083$.and y(x) = $x - \frac{x^3}{6}$. The closed solution of y(x) = sin x, which is the exact solution for larger values of m and it has been depicted in Fig 1.

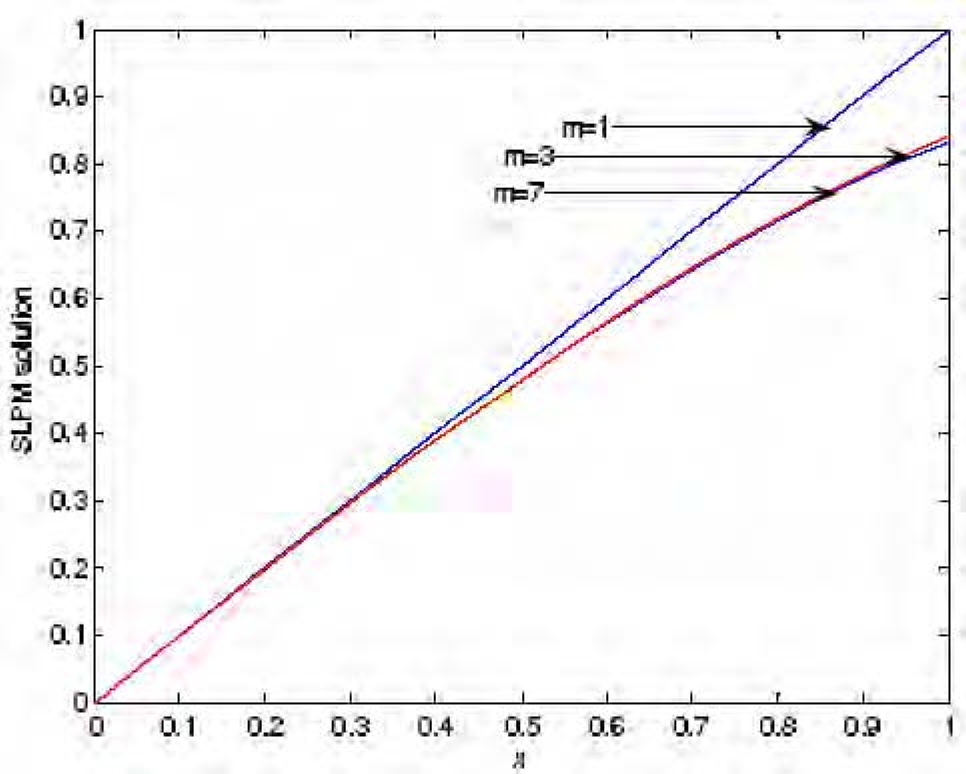

Fig 1: SLPM solution of Example 5 for different values of m

Example 6

Consider the nonlinear boundary value problem for the integro-differential equations related to the Blasius problem [49].

$$y''(x) = \alpha - \frac{1}{2}\int_0^x y(t)\, y''(t)\, dt, \quad -\infty < x < 0 \tag{19}$$

subject to the boundary conditions $y(0) = 1,\ y'(0) = 1$ and $y'(x) = 0$ *when* $x \to -\infty$

We solve (19) with m=6.

y(x)=$c_0\, p_0 + c_1\, p_1 + c_2\, p_2 + c_3\, p_3$

$= x + \frac{1}{2}\alpha x^2 - \frac{1}{48}\alpha x^4 - \frac{1}{240}\alpha^2 x^5 + \ldots$ which is the exact solution reported in [49]

VII. SALIENT FEATURES

The proposed method is very simple in application and SLPM the solution can be obtained in bigger interval. Unlike Adomian Decomposition Method(ADM), Homotopy Analysis Method(HAM) and Homotopy Perturbation Method(HPM), the SLPM do not require the Adomian polynomials, Lagrange multiplier, correction functional, stationary conditions and calculating integrals, which eliminate the complications that exists in the ADM, HAM and MHPM.

The solution obtained by means of SLPM is an infinite power series for appropriate initial approximation, which can be, in turn, expressed in closed form of exact solution. i.e. the infinite series solution is obtained for each problem by increasing the value of m, which in turn converges to closed form of exact solution, the error tends to zero and ensures stability.

VIII. CONCLUSION

In this work, we have proposed the shifted Legendre polynomials method (SLPM) for Solution of thelinear and nonlinear Volterra-Fredholm integral equations. The properties of shifted Legendre polynomials are used to reduce the problem to the solution of algebraic equations with appropriate coefficients which provide exact solutions for all the chosen problems. Moreover, the convergence analysis, error estimations clearly reveal its validity and potential use of applicability to any phenomena governed by this equation. In future, we may use this proposed method for solving other non linear fractional integro-differential equations and fractional partial differential equations also.

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