Calculating the Project Network Critical Path in Uncertainty Conditions

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Abstract— **Correct scheduling of the project is the necessary condition for the project success. In traditional models, the activities duration times are deterministic and known. In real world however, accurate calculation of time for performing each activity is not possible and is always faced with uncertainty. In this paper, the duration of each activity is estimated by the experts as linguistic variables and the said variables are represented in fuzzy numbers form using the fuzzy theory. Estimating the project accomplishment duration and determining the project critical path will be possible through resolving a fuzzy linear programming model. For solving model, Fuzzy Critical Path Method Algorithm (FCPMA) is introduced that uses fuzzy numbers ranking. In none of this method steps the defuzzification of the fuzzy numbers occurs, and the project accomplishment duration is gained in trapezoidal fuzzy number. Finally the performance of the introduced algorithm is shown using an application example.**

Keywords: Project scheduling, Critical path method, Fuzzy ranking number

I. INTRODUCTION

The project network is defined as a set of activities performed according to the precedence constraint of the activities. A network path is a path from the beginning node to the last node. The path length is equal to the total sum of the activities duration performed on the path.

The accomplishment duration of the project is equal to the length of the lengthiest network path which is called the critical path. The project is accomplished when all the activities existing in the critical path have been accomplished.

Unreal estimation of the project duration entails failure in estimating the project costs and planning the project resources, distrust of the clients and imposing fines in contracts, inconsistency in project progress reports and etc. Using inappropriate methods in estimating the activities duration causes unreal estimation of the project duration.

In the real world, the project is executed in an environment the uncertainty being one the principal features of it. One of these uncertainties in project planning process is estimating the activities duration. The duration of activities is usually estimated by the experts and considering their judgment and expertise. The experts use terms like “almost”, “a little more”, “about”, “more or less”, etc. These terms clearly show some kind of uncertainty. Some of methods exist for using this uncertainty in approximating the duration of the activities, the most important and employed of which being the probable methods like PERT and GERT.

These methods use probability distributions such as normal distribution and beta distribution for estimating the duration of the project activities. To use the probability distributions, the repeatable random samples are needed, which is not possible properly due to the unique activities of the project and their little antecedent. Moreover, when using a probability distribution, the scheduling variables are dependent upon the distribution treatment, causing restriction of the project scheduling. One basic solution for solving such problems is using the fuzzy theory. The fuzzy theory was first introduced by Prof. Zadeh in 1965. The fuzzy theory established a new attitude towards different sciences including the project scheduling. This is a way towards realizing the project scheduling models through considering the uncertainty in decision making parameters and using the experts’ mental models. Numerous researches have been carried out about fuzzy project scheduling.

Zielenski and Chanas [1] presented a natural generalization of the criticality notion in a network with fuzzy activity times. Dubois et al.[2] introduced a heuristic method for calculating the set of possible values from the latest starting times and float of the activities. Zielenski[3] used polynomial algorithms for determining the interval of the latest starting time in the network.

Feng and Jing [4] have presented a CPM fuzzy in which the a-cut has been used. Chen and Hsush [5] defined the most critical path and the relative path degree of criticality, which were theoretically sound and easy to use in practice.

In this paper the experts’ views described as language variables are shown in trapezoidal fuzzy numbers. Then the estimation of the fuzzy time required for accomplishment of the project and determining the fuzzy critical path of the project is performed through solving a fuzzy linear programming model.

The fuzzy number of $\tilde{A}$ is set in [0, 1] span by $\mu_{\tilde{A}}(x)$ membership function (Fig.1). In this research the fuzzy numbers have been considered in trapezoid form and a trapezoidal fuzzy number is shown as *(a_1, a_2, α, β)*. The trapezoid membership function is defined as the following:

$$\mu_{\tilde{A}}(x) = \begin{cases} 1 - \frac{a_1 - x}{\alpha} & a_1 - \alpha \le x \le a_1 \\ 1 & x \in [a_1, a_2] \\ 1 - \frac{x - a_2}{\beta} & a_2 \le x \le a_2 + \beta \end{cases}$$

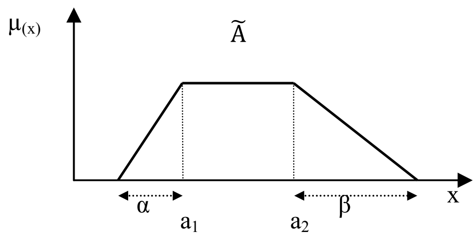

Fig. 1. Definition of $A = (a_1, a_2, \alpha, \beta)$

Suppose $\tilde{A} = (a_1, a_2, \alpha_A, \beta_A)$ and $\tilde{B} = (b_1, b_2, \alpha_B, \beta_B)$ are the two fuzzy numbers Summation of them is done as follows:

$$\tilde{A} + \tilde{B} = (a_1 + b_1,\ a_2 + b_2,\ \alpha_A + \alpha_B,\ \beta_A + \beta_B)$$

This paper is totally different in solving method compared with the previous researches. A resolving algorithm is introduced for solving the fuzzy linear programming model. This algorithm uses fuzzy numbers ranking based on the random processes. In this algorithm, the numbers are fuzzy from the beginning to the end and do not change to crisp numbers. The algorithm output is the critical path and the duration of the project accomplishment which is gained as a trapezoidal fuzzy number.

II. Problem formulation

In this research, the project is defined by a direct acyclic graph *G =(V,E)*, in which *V* (vertex) and *E* (edge) are the sets of nodes and arcs, respectively. Arcs and nodes represent activities and events, respectively. *G (V, E)* is demonstrated as a matrix *Am×n* in which m and n denote the number of nodes and arcs.

Matrix *Am×n* is called the node -arc incidence matrix for graph *G(V,E)* . Matrix *A* has one row for each node of the network and one column for each arc. Each column of *A* contains exactly two nonzero coefficients :"+1" and "-1". The column corresponding to arc *j* contains "+1" if *i* is the node starting arc *j*, "-1" if *i* is the node ending arc *j*, and "0" otherwise.

$$A = [a_{ij}] \quad i = 1, \dots, \dots, m \qquad j = 1, \dots, \dots n$$

$$a_{ij} = \begin{cases} 1 & \text{if node } i \text{ starts arc } j \\ -1 & \text{if node } i \text{ ends arc } j \\ 0 & \text{otherwise} \end{cases}$$

The aim is finding the project accomplishment duration which is the same as the critical path length. The following notation is used to describe the fuzzy linear programming model:

Parameters:

$\tilde{t}_j$: Fuzzy duration of activity *j* is shown as trapezoidal fuzzy number;

a_{ij}: The entry of incidence matrix, as defined before;

b_i: The available supply in node *i*;

Decision Variable:

$$x_j = \begin{cases} 1 & \text{if the acivity } j \text{ is in the path ;} \\ 0 & \text{otherwise.} \end{cases}$$

Objective functions:

$\tilde{T}$: Fuzzy duration of project complementation;

Fuzzy critical path is formulated considering the network matrix *Am×n* as follows:

$$\tilde{T} = \text{Max} \sum_{j=1}^{n} x_j * \tilde{t}_j \tag{1}$$

St:

$$\sum_{j=1}^{n} a_{ij} * x_j = b_i \qquad i = 1,2, \dots \dots \dots, m$$

$$b_i = \begin{cases} 1 & \text{if } i = 1 \\ -1 & \text{if } i = m \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

$$x_j = \in \{0, 1\} \qquad \forall j \tag{3}$$

"(1)" and its constraints calculate Fuzzy duration of critical path. (2)" defines the feasible path in the project network and is used for the precedence relations between activities. "(3)" is sign constrain.

III. The solution procedure

A resolving algorithm is introduced to solve the fuzzy linear programming model. However it is necessary to consider the fuzzy numbers ranking method. Different methods of ranking the fuzzy numbers have been suggested and discussed [6,7]. To solve the model it is necessary to use a method for ranking the objective values that denote the path lengths of the project network

A. Comparison of the fuzzy numbers

In this paper FNR algorithm is presented that uses fuzzy number ranking [7].Let suppose a fuzzy set or a fuzzy number like *M* as shown in Fig. 2.

$G(\lambda)$ is defined as follow:

$$G(\lambda) = \frac{\int_{\lambda}^{U} \mu(x)dx}{\int_{L}^{U} \mu(x)dx}$$

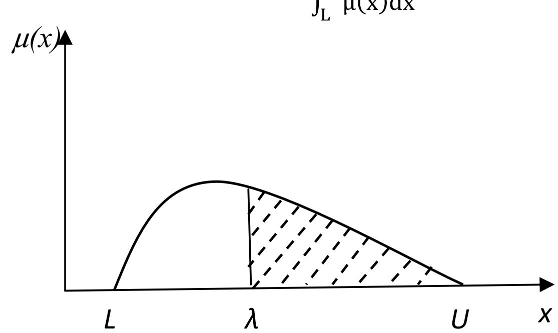

Fig. 2. G(λ) =The ratio of the shaded area over the total area[7]

To make a comparison between two fuzzy numbers, the $G(\lambda)$ value of them is calculated. Then it can be said the number that has greater $G(\lambda)$ is greater in lieu of the λ. Numerous $G(\lambda)$s are calculated as per numerous λs in *[L,U]* span. They are evaluated point by point and rank them at each point.

For each of the two numbers $\tilde{A}$, $\tilde{B}$, the $G(\lambda)$ value is compared as per any one of the λs. The same λ is assaigned to the number having greater $G(\lambda)$. This procedure is performed for numerous λs in $\lambda \in [L, U]$ span.

For example, there are two trapezoidal fuzzy numbers, $\tilde{A}$ and $\tilde{B}$ as shown in Fig.3 $G_{\tilde{A}}(\lambda)$ and $G_{\tilde{B}}(\lambda)$ are defined as follow:

$$G_{\tilde{A}}(\lambda) = \frac{\int_{\lambda}^{a_2+\beta_a} \mu_{\tilde{A}}(x)dx}{\int_{a1-\alpha_a}^{a2+\beta_a} \mu_{\tilde{A}}(x)dx}$$

$$G_{\tilde{B}}(\lambda) = \frac{\int_{\lambda}^{b_2+\beta_b} \mu_{\tilde{B}}(x)dx}{\int_{b1+\alpha_b}^{b2+\beta_b} \mu_{\tilde{B}}(x)dx}$$

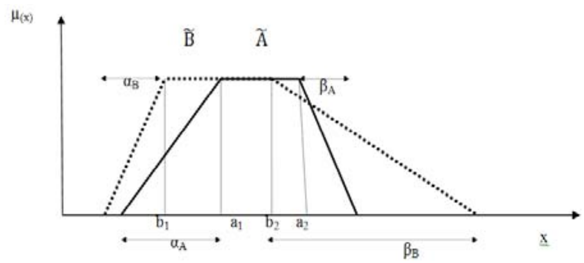

Fig .3. Definition of $\tilde{A} = (a_1, a_2, \alpha_A, \beta_A)$ and $\tilde{B} = (b_1, b_2, \alpha_B, \beta_B)$

$$G_{\tilde{A}}(\lambda) = \begin{cases} 1 - \frac{\lambda^2}{\alpha_A(\alpha_A+\beta_A+2a_1-2a_2)} & \text{if } a_1 - \alpha_A \le \lambda < a_1 \\ \frac{\beta_A+2(a_2-\lambda)}{\alpha_A+\beta_A+2(a_2-a_1)} & \text{if } a_1 \le \lambda \le a_2 \\ \frac{(a_2+\beta_A-\lambda)^2}{\beta_A(\alpha_A+\beta_A+2a_2-2a_1)} & \text{if } a_2 < \lambda \le a_2 + \beta_A \end{cases}$$

$$G_{\tilde{B}}(\lambda) = \begin{cases} 1 - \frac{\lambda^2}{\alpha_B(\alpha_B+\beta_B+2b_1-2b_2)} & \text{if } b_1 - \alpha_B \le \lambda < b_1 \\ \frac{\beta_B+2(b_2-\lambda)}{\alpha_B+\beta_B+2(b_2-b_1)} & \text{if } b_1 \le \lambda \le b_2 \\ \frac{(b_2+\beta_B-\lambda)^2}{\beta_B(\alpha_B+\beta_B+2b_2-2b_1)} & \text{if } b_2 < \lambda \le b_2 + \beta_B \end{cases}$$

L and U are defined as follow:

$$L = \min\big((a_1 - \alpha_A), (b_1 - \alpha_B)\big)$$
$$U = \max\big((a_2 + \beta_A), (b_2 - \beta_B)\big)$$

The following equation must be solved for calculation of the λ:

$$G_{\tilde{B}}(\lambda) = G_{\tilde{A}}(\lambda)$$

$$\frac{\int_{\lambda}^{b_2+\beta_b} \mu_B(x)dx}{\int_{b1+\alpha_b}^{b2+\beta_b} \mu_B(x)dx} = \frac{\int_{\lambda}^{a_2+\beta_a} \mu_A(x)dx}{\int_{a1-\alpha_a}^{a2+\beta_a} \mu_A(x)dx}$$

After calculation of the λ, the R_A and R_B can be calculated as follows:

$$R_A = \frac{\lambda - L}{U - L} \quad \text{and} \quad R_B = \frac{U - \lambda}{U - L}$$

By calculating RA and RB it can be said:

If $R_A > R_B$ then $\tilde{A} > \tilde{B}$
If $R_A = R_B$ then $\tilde{A} = \tilde{B}$
If $R_A < R_B$ then $\tilde{A} < \tilde{B}$

The FNR algorithm flowchart of comparison between two fuzzy numbers, $\tilde{A}$ and $\tilde{B}$ has been shown in Fig. 4.

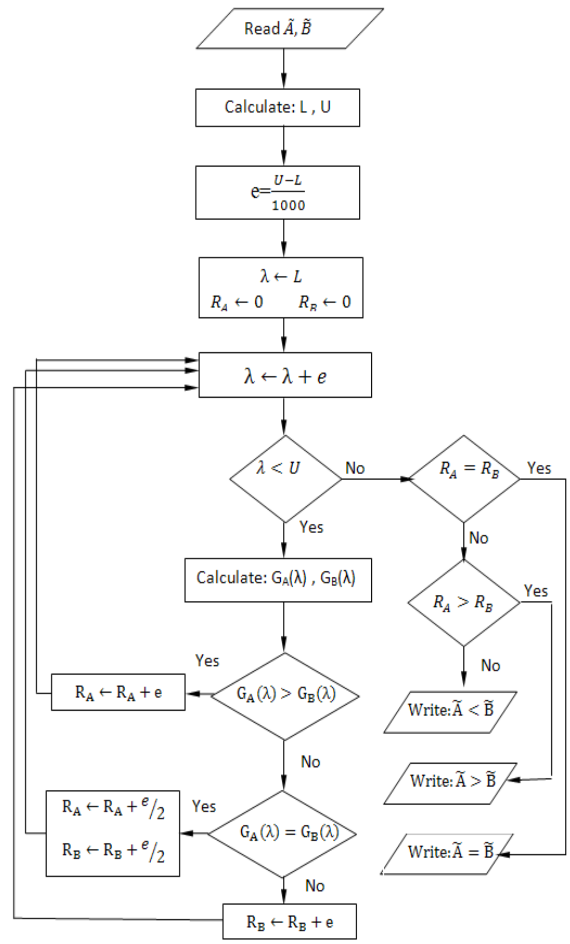

Fig. 4. Flowchart of FRN algorithm

B. *Calculating fuzzy critical path of the project*

In this paper, Fuzzy Critical Path Method Algorithm (FCPMA) is introduced for calculating the fuzzy critical path in which a path is shown using a binary string. The length of this string is equal to the number of the project activities.

$$[x_1, x_2, x_3, \ldots, x_n] \quad x_j \in [0, 1]$$

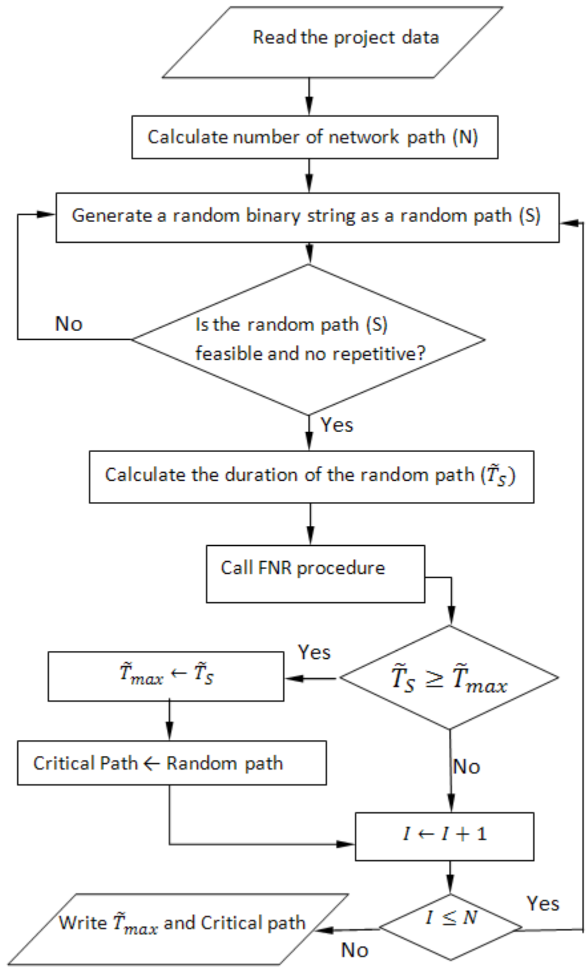

Fig. 5. Flowchart of FCPMA

The *FCPMA* is described as follows:

Step 1: The project data is read and number of the network's paths (*N*) is calculated. Then (*N*) number of non-repetitive feasible random paths is created. A network random path (*S*) must satisfy the model constraints to be feasible. These constraints are as follow:

$$\sum_{j=1}^{n} a_{ij} * x_{sj} = b_i \qquad i = 1,2 \dots, m \qquad s = 1,2, \dots, N$$

$$x_{sj} = \in \{0, 1\} \qquad j = 1,2, \dots, n \qquad s = 1,2, \dots, N$$

$$x_{sj} = \begin{cases} 1 & \text{if the acivity } j \text{ is in the path } s; \\ 0 & \text{otherwise.} \end{cases}$$

Step 2: Total project duration is calculated for each random feasible path (*S*). For example, in the project network (Fig. 5) the binary string of [0,1,1,0,0,0,0,0,1,0,0,1] is a random feasible path and the corresponding fuzzy duration is equal to $\tilde{t}_s$=(18 , 23 , 4 , 9).

$$\tilde{T}_s = \mathrm{Max} \sum_{j=1}^{n} x_{sj} * \tilde{t}_j \qquad s = 1,2, \dots, N$$

Step 3: The path having the longest duration is selected as the critical path from among the existing paths.

As the duration of the paths are fuzzy numbers, the algorithm FNR (Fig. 3) is used to make comparison among them. All the FCPMA steps have been reviewed in Fig. 5.

IV. Application Example

In this section an example having been solved by Chanas [1] and Chen [5] is resolved using the FCPMA and the results are compared with each other. This example is a project with 12 activities and 9 nodes (Fig. 6). the node -arc incidence matrix for project network in Fig. 7 and the activities' duration in trapezoidal fuzzy numbers have been described below.

$$\tilde{t}_1 = (1\,,1.5\,,1\,,1)\;,\tilde{t}_2 = (2\,,3\,,0\,,2)\,,\tilde{t}_3 = (6\,,7\,,0\,,2)\,,$$

$$\tilde{t}_4 = (0\,,0\,,0\,,0)\,,\tilde{t}_5 = (0\,,0\,,0\,,0)\;,\tilde{t}_6 = (2\,,3\,,1\,,2)\,,$$

$$\tilde{t}_7 = (9\,,9\,,1\,,1)\,,\tilde{t}_8 = (5\,,5\,,1\,,1)\,,\tilde{t}_9 = (4\,,4\,,2\,,2),$$

$$\tilde{t}_{10} = (3\,,4\,,2\,,0)\,,\tilde{t}_{11} = (8\,,9\,,2\,,4)\,,\tilde{t}_{12} = (6\,,9\,,2\,,3)$$

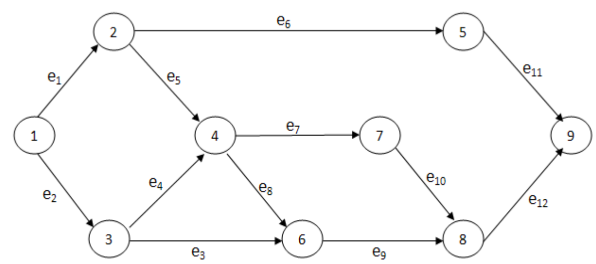

Fig. 6. Project network

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c} \begin{array}{cccccccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 & e_9 & e_{10} & e_{11} & e_{12} \end{array} \\
\begin{bmatrix}
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1
\end{bmatrix} \end{array}$$

Fig. 7. Network matrix

The above example has been solved by Chanas [1]. Table 1 is the answer gained from the Chanas model. Chanas has given the criticality level for different paths. The same example has been resolved by Chen [5]. Table 2 is the answer gained by the Chen model. Taking notice of the tables 1 and 2, the critical path of {1-2-3-4-7-8-9} is the answer of the problem.

TABLE I
THE PATH DEGREES OF CRITICALITY IN ABOVE EXAMPLE[1]

$1-2-5-9$	0.6269
$1-2-4-7-8-9$	0.5001
$1-2-4-6-8-9$	0.3854
$1-3-4-7-8-9$	1
$1-3-4-6-8-9$	0.0001
$1-3-6-8-9$	0.9941

TABLE II
THE RELATIVE PATH DEGREE OF CRITICALITY IN DESCENDING ORDER[5]

Path	$\boldsymbol{deg_{cr}^{p}(p_i)}$
$p_1 = \{1 \to 3 \to 4 \to 7 \to 8 \to 9\}$	1
$p_2 = \{1 \to 3 \to 6 \to 8 \to 9\}$	0.9574
$p_3 = \{1 \to 2 \to 4 \to 7 \to 8 \to 9\}$	0.9212
$p_4 = \{1 \to 3 \to 4 \to 6 \to 8 \to 9\}$	0.8789
$p_5 = \{1 \to 2 \to 4 \to 6 \to 8 \to 9\}$	0.8001
$p_6 = \{1 \to 2 \to 5 \to 9\}$	0.5717

To solve the above example using the FCPMA, the said method was first coded in Excel using VBA. Then the data gained from the above example were entered into the application software. After execution of the application, the project accomplishment duration was attained equal to (20, 25, 5, 6) and the binary string of [0,1,0,1,0,0,1,0,0,1,0,1] was gained as the critical path. This binary string corresponds to [e_2-e_4-e_7-e_{10}-e_{12}] critical path and this is exactly same {1-2-3-4-7-8-9}.

Comparison of the above three methods indicates that the gained critical path is the same in all the three methods, but the project accomplishment duration is a trapezoidal fuzzy number in FCPMA.

In FCPMA calculations all the project activities duration are in fuzzy number form and do not change into the crisp numbers in any steps. So the project accomplishment duration is also a fuzzy number.

As the activities' duration has been described in language variables form, the project manager expects the project accomplishment duration to be a fuzzy number. It can be said that the FCPMA is closer to the reality and entails less approximation in project accomplishment duration calculations.

V. CONCLUSION

The critical path identification and project accomplishment duration calculation is of important project manager tasks. The project activities duration is described in language variables form by the experts. In this paper, the language variables have been changed into fuzzy trapezoidal numbers using fuzzy theory. For fuzzy critical path calculations it is necessary to compare the fuzzy numbers with each other. In this paper, the FNR algorithm was used for comparison among the fuzzy numbers.

Then the FCPMA and its solution method were introduced using the FNR algorithm. Finally, an example was resolved using the FCPMA and the answer was compared with the answer gained from the presented method in [1] and [5]. As all numbers in FCPMA are fuzzy numbers and do not change into crisp numbers during calculations and the final answer also is fuzzy number, it can be said that the FCPMA is closer to the reality compared with other methods of resolving the fuzzy critical path.

The introduced method in this paper can be developed for the GERT networks as the future work.

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