Efficient Quadrature Operator Using Dual-Perspectives-Fusion Probabilistic Weights

Ashok Sahai & Sameer Verma [1]

Dept of Mathematics & Computer Science; The University of the West Indies, Trinidad & Tobago.

[1] Dept. of Information Systems; SFSU (San Francisco State Univ.; USA).

Abstract. A new quadrature formula has been proposed which uses weight functions derived using a probabilistic approach, and a rather-ingenious 'Fusion' of two dual perspectives. Unlike the complicatedly structured quadrature formulae of Gauss, Hermite and others of similar type, the proposed quadrature formula only needs the values of integrand at user-defined equidistant points in the interval of integration. The weights are functions of the impugned variable in the integrand, and are not mere constants. The quadrature formula has been compared empirically with the simple classical method of numerical integration using the well-known "Bernstein Operator". The percentage absolute relative errors for the proposed quadrature formula and that with the "Bernstein Operator" have been computed for certain selected functions and with different number of node points in the interval of integration. It has been observed that the proposed quadrature formula produces significantly better results.

Keywords: *Probability; Percentage Absolute Relative Errors; Quadrature Formula; Weight Functions.*
AMS Classification: 41A10 & 41A55.

1. Introduction. Classically ([1], and [2]), even the celebrated Bernstein polynomial approximation operator has been used for numerical integration. In a rather simple set-up, such similar quadrature methods were of the form

$$\int_a^b f(x).dx \approx \sum_{i=1}^{i=n} w_i.f(x_i).$$

Where $\mathbf{x_i}$**'s** are equally spaced nodes & $\mathbf{w_i}$**'s are** the respective weights. For example:

- **Trapezoid:** $\int_a^b f(x).dx$ **≈ [(b – a)/2]*f (a) + [(b – a)/2]*f (b).**
- **Simpson:** $\int_a^b f(x).dx$ **≈ [(b – a)/6]*f (a) + [2(b − a)/3]*f ((a + b)/2) + [(b – a)/6]*f (b).**
- Similarly, for higher order polynomial, **Newton-Cotes rules**.

We note here one known thing already from interpolation that the equally-spaced nodes result in wiggle.

What other choice do we have? We might do well here to recall how we fix wiggle in interpolation: by moving the location of the nodes. Therefore, we might do better, possibly, by:

- Freeing ourselves from equally spaced nodes.
- Combining selection of the nodes and selection of the weights into one quadrature rule.

**The Corresponding Author's Email~* sahai.ashok@gmail.com.

Gaussian quadrature chooses the points for evaluation in an optimal, rather than the equally- spaced way. The nodes $x_1, x_2, \ldots, x_n$ in the interval [a, b] and coefficients $c_1, c_2, \ldots, c_n$ are chosen to minimize the expected error in **:**

$$\int_a^b f(x).dx \approx \sum_{i=1}^{i=n} c_i . f(x_i).$$

This equation needs **'n'** parameters. The class of polynomials of degree **n-1** has **n** parameters, so for this class of polynomials we could expect our formula to be exact.

For example, we want to determine $\mathbf{c_1, c_2, x_1,}$ **and** $\mathbf{x_2}$ so that the formula ~

$$\int_{-1}^{1} f(x).dx \approx \mathbf{c_1.f(x_1) + c_2.f(x_2).}$$

gives the exact result whenever f(x) is a polynomial of degree 3. We can substitute **1, x,** $\mathbf{x^2, x^3}$ **for f(x).** This gives the following equations: $\mathbf{c_1 + c_2} = \int_{-1}^{1} f(x).dx$ **= 2;** $\mathbf{c_1.x_1 + c_2.x_2} = \int_{-1}^{1} f(x).dx$ **= 0;** $\mathbf{c_1.(x_1)^2 + c_2.(x_2)^2} = \int_{-1}^{1} x^2.dx$ **= 2/3, &** $\mathbf{c_1.(x_1)^3 + c_2.(x_2)^3} = \int_{-1}^{1} x^3.dx$ **= 0. This has unique solution:** $\mathbf{c_1 = c_2 = 1, x_1 = -\sqrt{3}/3, x_2 = -x_1}$**; meaning that→**

$$\int_{-1}^{1} f(x).dx \approx \mathrm{f}\,\mathbf{(-\sqrt{3}/3)} + \mathrm{f}\,\mathbf{(\sqrt{3}/3).}$$

It is interesting that this formula is exact for every polynomial of degree 3 or less. This technique can be used to determine parameters for higher n, but there is an easier method using the Legendre polynomials. Nevertheless, in the context of our proposition, we can note here an important fact that **"√3/3" is an irrational number,** and hence the exact calibration of the value of "f **(√3/3)**" is not feasible. That value, therefore, would be subject to some degree of approximation. This leads us to the following remark, in the context of our proposition, **"Unlike complicatedly structured quadrature formulae of Gauss, Hermite and others of similar type, the proposed quadrature formula only needs the values of integrand at user-defined equidistant points in the interval of integration".**

The preceding fact motivated us to ponder-n-wonder as to whether or not we would be able **to retain the simplicity of "equidistant points in the interval of integration" in our proposed quadrature formula**, and still aspire successfully for its optimality. In fact we were successful. The next section details the **'How'** part of it.

And the section following that numerically illustrates that the potential of this new/ proposed operator was so profound as to make it almost the 'Optimal' one, inasmuch as it excelled over the well-known powerful "Gaussian Quadrature", besides being exceedingly superior to the famous "Bernstein Operator Quadrature". The "Bernstein Operator" happened to be the foundational structure for the build-up of the proposed one, with the used weight-functions therein derived pursuing a **probabilistic approach**, and **a rather-ingenious 'Fusion' of two dual perspectives**. We are going to detail this in following section.

2. The Proposed Probabilistically Weighted Qudrature Operator.

The problem of approximation arises in many contexts of 'Numerical Analysis and Computing', and 'Quadrature' is one such. Weierstrass (1885) proved his celebrated approximation theorem: if f ε C [a, b]; for every δ > 0; there is a polynomial 'p' such that

$\|f - p\| < \delta$.

In other words, the result established the existence of an algebraic polynomial, in the relevant variable, capable of approximating the unknown function in that variable, as closely as we please! This result was a big beginning of the mathematicians' interest in 'Polynomial Approximation' of an unknown function using its values generated experimentally or known otherwise at certain chosen 'Knots' of the domain of the relevant variable, as of interest to the scientist concerned. The Great Russian mathematician S. N. Bernstein proved the Weierstrass' theorem in a manner which was very stimulating and interesting in many ways.

He first noted a simple but important fact that if the Weierstrass' theorem holds for C [0, 1], it also holds for C [a, b] and holds conversely. Essentially C [0, 1] and C [a, b] are identical, for all practical purposes; as they are linearly isometric as normed spaces, order isomorphic as algebras (rings). Most important contribution in the Bernstein's proof of this theorem consisted in the fact that Bernstein actually displayed a sequence of polynomials that approximate a given function f(x) ε C [0, 1]. If f (x) is any bounded function on C [0, 1], the sequence of Bernstein's Polynomials for f (x) is defined by:

(Bn (f)) (x) = $\sum_{k=0}^{k=n} w_{k,n} * f\left(\frac{k}{n}\right)$

(2.1)

Wherein, $w_{k,n} = \binom{n}{k} x^k (1-x)^{n-k}$ are the respective weights for **the values '$f\left(\frac{k}{n}\right)$' of the function at the knots"$\left(\frac{k}{n}\right)$" [k = 0 (1) n]**.

Thus if we assume that, **without any loss of generality** [subject to the change of origin-and-scale of the impugned-variable, if necessary], our interest is that in **developing the "Quadrature Operator" for the numerical integration of the definite integral** $\int_0^1 f(x)dx$**;** **(2.2)**

"f (x)" being a bounded unknown function with known/ knowable values at equi-distant knots **"$\left(\frac{k}{n}\right)$" [k = 0 (1) n].** Hence, one straightforward "Quadrature" using the Bernstein's Operator:

$$\int_0^1 f(x)dx = \sum_{k=0}^{k=n} w_{k,n} * f\left(\frac{k}{n}\right)$$

(2.3) Now, we are set to, first, launch the "probabilistic perspective" seminal to the proposition of our novel 'Quadrature Operator" in the same "Equi-Distant Knots" 'Set-Up' of Bernstein's Operator, in what follows.

Consider the fact that f (x) is bounded **unknown function in its domain C [0, 1]**. Let us, **call this impugned interval-line by O1- O2; "O1" and "O2" being the two end-points of this original-line C [0, 1].** Now consider **the line vis-à-vis the [0, 2],** the double of the original interval-line; say **call it by O1- O3, so that O2 is the middle point of this double-of-the-impugned-interval-line**. **Consider a random-point 'x' sitting on this double-line. This point x $[0 \leq x \leq 1]$ divides the double-line in two parts of lengths (1- x): (1+ x) :: O1 → x: x→ O3. This random-point [rational or irrational] is such that $k/n \leq x \leq (k+1)/n$. Now, probabilistically, the probability of k points/ knots being on the left of 1- x, and n – k being on its right could easily be seen to be:**

$$[\binom{n*(1-x)}{k}*\binom{n*(1+x)}{n-k}]/\binom{n}{n} \equiv$$

$$\equiv$$

$$[\binom{n*(1-x)}{k} * \binom{n*(1+x)}{n-k}] = \mathrm{P1}(x;k,n), \text{Say;}$$

[As $\binom{n}{n}$ = 1].

(2.4)

Reverting back to the original-interval **C [0, 1]** from the present double-interval **C [0, 2]** is simply accomplished by a division by **2: [(1-x) + (1+x)]/2 ≈ 1 ≈ (1-x)/2 + (1+x)/2~ Say, Pw1 (x; k, n) =**

$$=[\binom{n*(1-x)/2}{k} * \binom{n*(1+x)/2}{n-k}]\,. \qquad \textbf{(2.5)}$$

It is worth noting here that these **'Primal Probability-Weights' "Pw1(x; k, n)" are the typical coefficients of θ^n in the expansion of the identity involving binomials-product:**

$[(1+\theta)^{n*(1-x)/2}] * [(1+\theta)^{n*(1+x)/2}] = (1+\theta)^n$

Therefore, equating these coeffients, on both sides of this identity, and equating we get the following equation: **[As $\binom{n}{n}$ = 1]**

$$\sum_{k=0}^{k=n}[Pw1(x;k,n)] =$$

$$\sum_{k=0}^{k=n}[\binom{n*(1-x)/2}{k} * \binom{n*(1+x)/2}{n-k}] = 1.$$

(2.6)

Similarly, but in a **"dual" sense**, we could have a **visualization wherein this point x $[0 \leq x \leq 1]$ divides the double-line in two parts of lengths (1+ x): (1- x):: O1→ x: x→ O3. Now, similarly-n-analogously-probabilistically, the probability of k points/ knots being on the left of 1+ x, and n – k being on its right could easily be seen to be:**

$$[\binom{n*(1+x)}{k}*\binom{n*(1-x)}{n-k}]/\binom{n}{n} =$$

$$[\binom{n*(1+x)}{k} * \binom{n*(1-x)}{n-k}] = \mathrm{P2}(x;k,n), \text{Say;}$$

As $\binom{n}{n}$ = 1.

(2.7)

Reverting back to the original-interval C [0, 1] from the present double-interval is simply accomplished by a division by **2: [(1-x) + (1+x)]/2 ≈ 1 ≈ (1+x)/2 + (1-x)/2~ Say, P_w2 (x; k, n) =**

$$= [\binom{n*(1+x)/2}{k} * \binom{n*(1-x)/2}{n-k}]$$

(2.8)

It is worth noting here that these **'Dual Probability-Weights' "Pw2(x; k, n)" are typical coefficients of θ^n in the expansion of the binomials-product:**

$[(1 + \theta)^{n*(1+x)/2}] * [(1 + \theta)^{n*(1-x)/2}] = (1 + \theta)^n$

Therefore, equating these coeffients, on both sides of this identity, and equating we get the following equation: **[As $\binom{n}{n}$ = 1],**

$$\sum_{k=0}^{k=n}[Pw\,2(x;k,n)] =$$

$$= [\binom{n*(1+x)/2}{k} * \binom{n*(1-x)/2}{n-k}] = 1.$$

(2.9)

Both'$Pw2(x;k,n)$' and '$Pw1(x;k,n)$', **dual to each-other, define 'Probability Distribution' each with the 'n+1' knots 'k/n' [k = 0 (1) n] being its support in C [0, 1].**

As such, now, **we define our 'Optimal Weights'; Say 'W_{opt} (x; k, n)' through a simple 'Fusion' of the two 'Probability Distributions' above in (2.6), and in (2.9) by taking their respective arithmetic mean.**

Therefore, we have:

$$W_{opt}(x;\ k,\ n) = \{ [\binom{n*(1-x)/2}{k} * \binom{n*(1+x)/2}{n-k}] + [\binom{n*(1+x)/2}{k} * \binom{n*(1-x)/2}{n-k}] \}/2$$

(2 .10)

Thence, we propose a '**new quadrature formula'** which uses weight-functions derived **using a probabilistic approach, and a rather-ingenious 'Fusion' of two dual perspectives,** as above. **This 'New Optimal Probabilistic Quadrature Operator' happens to be as below.**

Say; (On (f)) (x) =

$$\sum_{k=0}^{k=n} Wopt(x;\ k,n) * f(\frac{k}{n})$$

(2.11)

Wherein, "W_{opt} (x; k, n)" is as defined in (2.10).

At this point, we note that original interval **C [0, 1]** is the domain of the impugned bounded function 'f (x)'. Also, we note that **the intervals $[\frac{1}{2} - \frac{x}{2}, \frac{1}{2}]$ & $[\frac{1}{2}, \frac{1}{2} + \frac{x}{2}]$ are two sub-intervals of this domain.** **It is significant to observe that, with x ε [0, 1], whereas the former sub-interval will grow to become [0, $\frac{1}{2}$] for x=1, the later sub-interval will grow to become [$\frac{1}{2}$, 1] for x=1. On the other hand, for x=0, both of them will degenerate to the point "$\frac{1}{2}$"! In a sense, the two intervals are not only complimentary to each-other, but are dual to each-other.**

It is interesting to note that if we use n [n being a positive integer] points/ knots, beside '0' on C [0, 1], our proposed operator "(On (f)) (x)" has 'Zero-Error'

for the "Quadrature of f (x) = x^m ", ∀ m $\in [o(1)n], i.e. \forall\, m \leq n.$

Another point of curiosity would be to discover as to how our proposed operator "(On (f)) (x)" performs vis-à-vis the well-known 'Gaussian Quadrature Operator'. To accomplish that, and as an example of important applications we take to the "Quadrature Problem" in the context of 'Normal Distribution's Area' calculations. As the Normal/ Gaussian distribution is a 'Symmetric' one, it suffices to consider only the right-tale probability-areas, say, for 'Standard Normal Distribution ~ N (0, 1).

Let us consider the calculation of area under the right-tail up to 2.5 (say 'p = 2.5'):

$$\int_0^1 [p.\exp(-p^2 * x^2)/\sqrt{(2*\pi)}]\ dx = (1/2) \cdot \{\int_{-1}^1 [p.\exp(-p^2 * x^2)/\sqrt{(2*\pi)}]\ dx\}. \qquad (2.12)$$

While we would target the evaluation of the left-hand-side expression in (2.12) above for using our proposed operator "(On (f)) (x)", we would use the well-known 'Gaussian Quadrature Operator' for the evaluation of the right-hand-side expression in (2.12), above.

We did so using 'Maple 12 Evaluation-Version' ([3]) for n = 2, 3, 4, and 5, respectively for both of these operators.

The actual value of the relevant quadrature/ numerical definite integral in (2.12) was found to be **"0.4937903345". Results [Quadrature 'Estimated Values' using 'n' points for the operators [our operator "(On (f)) (x)" & that for Gaussian Quadrature "(Gn (f)) (x)", and their respective 'Percentage Relative Errors' ~ Say; "EstValue (On (f)) (x)" & "EstValue (Gn (f)) (x)"/ "PRError (On (f)) (x)" & "PRError (Gn (f)) (x)" for the n-values] are reported in the following table.**

Table 2.1 [Vis-à-vis 'Actual value: "0.4937903345"].

Quantity-Name↓	**n = 2**	**n = 3**	**n = 4**	**n = 5**
EstValue (On (f)) (x)	0.4149387746	0.5079701874	0.4910622648	0.4945175189
EstValue (Gn (f)) (x)	0.3519329982	0.5282411175	0.4874191154	0.4947504974
PRError (On (f)) (x)	15.968631700	2.871634358	0.5524753138	0.1472658230
PRError (Gn (f)) (x)	28.728252940	6.976803836	1.2902680860	0.1944474877

3. Numerical Study

This section is also of prime interest, as herein we try to illustrate the potential of our proposed probabilistic operator **"(On (f)) (x)".** As apparent in the second section, as a prelude to our proposed operator, **the mother-operators are "Bernstein's Polynomials for f (x) (Bn (f)) (x)" in (2.1).** As such, we could not have an idea about their relative supremacy in terms of better estimation potential; we have to discover their relative supremacy of efficient estimation only **via a 'Numerical Study'**, as attempted in what follows in this section. **In this simulated numerical study we have chosen four illustrative example-functions: exp (x), sin (2+x), 10^x, and ln (2+x)**. For simplicity of the numerical illustration we have confined to **only three** chosen **illustrative n-values to be 3, 6, and 9.**

We have considered numerical values (per the illustrative numerical study) of the **"Percentage Relative Absolute Errors"** in using the relevant operators by the evaluation of the expressions: **namely, say, "PRAbsErr (●) (In %)"** for **"(On (f)) (x)"; &** for **"(Bn (f)) (x)",** respectively→

$$[\left|\int_0^1 O_n(f;x) - \int_0^1 f(x)dx\right| \text{ x } 100]/\ [\int_0^1 f(x)dx] =$$

=PRAbsErr (O_n); Say. And

$$[\left|\int_0^1 B_n(f;x) - \int_0^1 f(x)dx\right| \text{ x } 100]/\ [\int_0^1 f(x)dx] =$$

=PRAbsErr (B_n); Say.

These **"Percentage Relative Absolute Errors"** [~ **"PRAbsErr (●) (In %)"**], calculated using the ***"MAPLE 12[Evaluation-Version]"*** **code,** are tabulated in the following tables in the **"APPENDIX". This illustration has amply supported the fact that the "Percentage Absolute Errors' Numerical-Values" for our proposed probabilistic operator "(On (f)) (x)" are significantly lower than those for "Bernstein's Polynomials for f (x); (Bn (f)) (x)" in (2.1).**

References:

[1]. E. W. Cheney and A. Sharma, Bernstein power series, Canad. J. Maths. 16 (1964), 241- 252.

[2]. P. P. Korovkin, Linear Operators And Approximation Theory. Hindustan Publishing, Delhi, 1960.

[3]. http://www.maplesoft.com/contact/webforms/maple_evaluation.aspx ~ Maple 12 Evaluation-Version.

APPENDIX.

Table 3.1.

Percentage Relative Absolute Error [~ "PRAbsErr (●) (In %)] of the Operators (In %) for Example-Function: f(x) = exp(x).

Operator↓	For n = 3↓	For n = 6↓	For n = 9↓
PRAbsErr (O_n)	0.01503391328000000	4.65581365600*10^(-7)	0.00000000000000000
PRAbsErr (B_n)	2.74257954800000000	1.3694167400000000	0.91234061520000000

Table 3.2.

Percentage Relative Absolute Error [~ "PRAbsErr (●) (In %)] of the Operators (In %) for Example-Function: f(x) = 10^x.

Operator↓	For n = 3↓	For n = 6↓	For n = 9↓
PRAbsErr (O_n)	0.3801436230000000	0.00004244431855000	0.9123406152*10^(-7)
PRAbsErr (B_n)	13.824606900000000	6.86560988700000000	4.56173511600000000

Table 3.3.

Percentage Relative Absolute Error [~ "PRAbsErr (●) (In %)] of the Operators (In %) for Example-Function: f(x) = sin (2+x).

Operator↓	For n = 3↓	For n = 6↓	For n = 9↓
PRAbsErr (O_n)	0.015850934550000000	3.485257690*10^(-8)	0.0000010804298840
PRAbsErr (B_n)	2.814637353000000000	1.4093066240000000	0.9401730422000000

Table 3.4.

Percentage Relative Absolute Error [~ "PRAbsErr (●) (In %)] of the Operators (In %) for Example-Function: f(x) = ln (2+x).

Operator↓	For n = 3↓	For n = 6↓	For n = 9↓
PRAbsErr (O_n)	0.002738167797000000	0.000001044481147000	0.000001429289991000
PRAbsErr (B_n)	0.002738167797000000	0.250934066800000000	0.167179157600000000

ISSN : 0975-4024