Eigenvalues of tridiagonal matrix using Strum Sequence and Gerschgorin theorem

T.D.Roopamala
Department of Computer Science and Engg.,
Sri Jayachamarajendra College of Engineering
Mysore INDIA
roopa_td@yahoo.co.in

S.K.Katti
[2]Research Supervisor S..J.C.E. Mysore University, Mysore City, Karnataka INDIA
Mysore University, Mysore City, Karnataka INDIA
skkatti@indiatimes.com

Abstract: **In this paper, computational efficient technique is proposed to calculate the eigenvalues of a tridiagonal system matrix using Strum sequence and Gerschgorin theorem. The proposed technique is applicable in various control system and computer engineering applications.**

Keywords- Eigenvalues, tridiagonal matrix, Strum sequence and Gerschgorin theorem.

I.INTRODUCTION

Solving tridiagonal linear systems is one of the most important problems in scientific computing. It is involved in the solution of differential equations and in various areas of science and engineering applications such control system [1] and computer science [2, 3]. It is also occur in a wide variety of applications, such as the construction of certain splines and the solution of boundary value problems. There are various numerical techniques available in the literature, which are useful for determining eigenvalues of real symmetric matrices [4]. In most of these methods, given system matrix is converted into tridiagonal form. There are various methods also given in the literature for determining eigenvalues of a tridiagonal matrix [4]. In this method, strum sequence and bisection method is used to determine the eigenvalues of a given real symmetric triangular matrix. It is observed that it require large number of iterations to compute the eigenvalues. It is observed that these iterations can be reduced by using well known Gerschgorin theorem [4]. In [5], technique is presented to identify the eigenvalues on the right half the s-plane using Gerschgorin theorem [4]. The extension to this approach is given in [6]. One of the leading methods for computing the eigenvalues of a real symmetric matrix is Given's method [4]. In that method, after transforming the matrix into diagonal form say, $'S'$, the leading principal minors of $\left|S-\lambda I\right|$ form a strum sequence. Then, using bisection approach, change of sign in various strum sequence is observed. Further, based on this, eigenvalue can be determined by repeatedly using bisection method. In [7], various applications are presented based on Gerschgorin theorem. Here, in this paper, similar to these existing applications, we have used Gerschgorin theorem in Strum sequence to determine eigenvalues in computationally efficient manner. In order to show the comparative result, we have considered the example which was earlier considered in [4].

II. Givens Method for Symmetric Matrices [4]:

Let A be a real, symmetric matrix. The Givens method uses the following steps:

(a) Reduce A to a tridiagonal form using plane rotations

(b) Form a strum sequence, study the changes in sign in the sequences and find the eigenvalues. The reduction to a tridiagonal form is achieved by using the orthogonal transformation. Suppose the orthogonal matrix is given as

$$[B] = \begin{bmatrix} b_1 & c_1 & 0 & 0 & \dots & 0 \\ c_1 & b_2 & c_2 & 0 & \dots & 0 \\ 0 & c_2 & b_3 & c_3 & \dots & 0 \\ . & . & . & . & . & \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & 0 & c_{n-2} & b_{n-1} & c_{n-1} \\ 0 & 0 & 0 & 0 & c_{n-1} & b_n \end{bmatrix} \quad (1)$$

The number of plane rotations required to bring a matrix of order n to its tridiagonal form is $\frac{1}{2}(n-1)(n-2)$. We know that A and B have the same eigenvalues. If $c_i \neq 0, i = 1,\dots,n-1$ then, the eigenvalues are distinct.

Now, we define

$f_n = |\lambda I - B| =$

$$\begin{bmatrix} \lambda - b_1 & -c_1 & 0 & \dots & 0 & 0 & 0 \\ -c_1 & \lambda - b_2 & -c_2 & \dots & 0 & 0 & 0 \\ . & . & . & \dots & 0 & 0 & 0 \\ . & . & . & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & -c_{n-2} & \lambda - b_{n-1} & -c_{n-1} \\ 0 & 0 & 0 & \dots & \dots & -c_{n-1} & \lambda - b_n \end{bmatrix} \quad (2)$$

Expanding by minors, the sequence $\{f_n\}$ satisfies

$$f_0 = 1,\ f_1 = \lambda - b_1 \quad (3)$$

and

$$f_r = (\lambda - b_r) f_{r-1} - c_{r-1}^2 f_{r-2};\ 2 \le r \le n \quad (4)$$

If none of the $c_1, c_2, ..., c_{n-1}$ vanish, then $\{f_n\}$ is a Strum sequence. That is, if $V(x)$ denotes the number of changes in sign in the sequence for a given number x, then the number of zeros of f_n in $[a,b]$ is $V(a)-V(b)$, provided a or b is not a zero of f_n. In this way, one can approximately compute the eigenvalues and by repeated bisections, one can improve these estimates. This method is explained in many text books on numerical analysis.

III. Proposed approach

In this strum sequence approach, selection of 'λ' is very much important. But in exiting approach, selection of λ is random approach, hence existing approach needs more computations to compute the eigenvalues. In such cases, Gerschgorin theorem [4] will be useful. It stated as follow:

Let P_k be the sum of the moduli of the elements along the k^{th} row excluding the diagonal elements a_{kk}. Then every eigenvalue of A lies inside or on the boundary of at least one of the circles

$$\left|\lambda - a_{kk}\right| = P_k, k = 1, ..., n. \tag{5}$$

Using this theorem, we obtain bounds on the real axis. The beauty of these bounds is such that all the eigenvalue lie between these bounds. So, we consider Gerschgorin bound 'λ' as initial approximation in this strum sequence method. Suppose, these bounds are denoted by E and D. Bound E always remains to the right side of D on the real axis in the s-plane whatever may be the value of E and D [6].

Thus, equation (3) and (4) modified as

$$f_0 = 1, \; f_1 = D - b_1 \tag{6}$$

and

$$f_r = (D - b_r) f_{r-1} - c_{r-1}^2 f_{r-2}; 2 \le r \le n \tag{7}$$

IV. Numerical Example

Now we calculate the eigenvalues of a given tridiagonal matrix as given below [4].

$$[A] = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \tag{8}$$

In [4], the eigenvalues are calculated using conventional method. It as follows:

4.1 Conventional method:

Step 1: Assume $\lambda = -1$, then from eq. (6) and (7), the strum sequence becomes

$$\begin{aligned} & f_0 = 1, \; f_1 = \lambda - 2; \\ & f_2 = (\lambda - 2) f_1 - f_0 = (\lambda - 2)^2 - 1 \\ & f_3 = (\lambda - 2) f_2 - f_1 = (\lambda - 2)^3 - 2(\lambda - 2) \end{aligned} \tag{9}$$

Step 2: Using strum sequence approach, we formulate the following array to get the eigenvalues.

Table 1: Existing approach of calculating eigenvalues using strum sequence

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
-1	+	-	+	-	3
0	+	-	+	-	3
1	+	-	0	+	2
2	+	0	-	0	-
3	+	+	0	-	1
4	+	+	+	+	0

From above table, there exist an eigenvalue, at λ=2 between in (0, 1) and also eigenvalue between (3, 4). We calculate actual eigenvalue by repeated bisection method. The exact eigenvalue is calculated as 0.585786. Similarly, by increasing the value of λ, we can estimate other eigenvalues of the matrix.

4.2 Proposed approach

Step 1: By applying Gerschgorin theorem to above matrix, we calculate bound as D=0 and E=4. It is shown in Fig. 1.

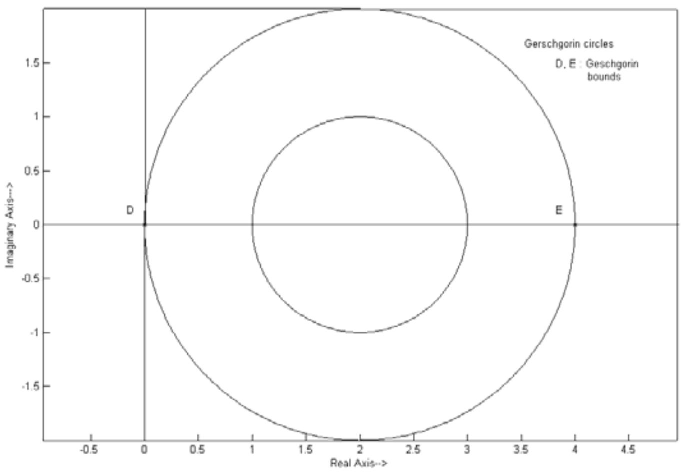

Fig1. Gerschgorin circles and bounds of a system matrix

Step 2: Using strum sequence approach, using eq. (6) and eq. (7), we formulate the following array to get the eigenvalues.

Table 2: Proposed approach of calculating eigenvalue using using strum sequence in the interval (0, 1)

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
0	+	-	+	-	3
1	+	-	0	+	2

From above table, there exist an eigenvalue, between in (0, 1). We calculate actual eigenvalue by repeated secant method. Similar to above approach, the exact eigenvalue is calculated as 0.585786.

Table 3: Proposed approach of calculating eigenvalue using using strum sequence in the interval (1, 3)

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
1	+	-	0	+	2
2	+	0	-	0	-
3	+	+	0	-	1

From the above table there exist an eigenvalue, is at λ =2

Table 4: Proposed approach of calculating eigenvalue using using strum sequence in the interval (3, 4)

λ	f_0	f_1	f_2	f_3	$V(\lambda)$
3	+	+	0	-	1
4	+	+	+	+	0

From above table, there exist an eigenvalue, between in (3, 4). We calculate actual eigenvalue by repeated secant method. Similar to above approach, the exact eigenvalue is calculated as 0.3.414214. From above table 1 2, 3 and 4 it is concluded that proposed approach is computationally efficient in comparison to existing approach.

Total Computation:

Existing method:

Bisection method: 855

Secant method: 225

V.CONCLUSION

In this paper, a computationally efficient approach is proposed for determination of eigenvalues of a given real symmetric triangular matrix using Gerschgorin theorem. In the existing method as we observe from the table the initial values are randomly chosen and the strum sequence method is applied to find the existence eigenvalues. Whereas in the proposed approach the strum sequence methods are applied at the point of intersection of the circles to decide the existence of eigenvalues in the Gerschgorin bound and then repeated secant method has been applied in these intervals to compute the exact eigenvalues. The secant method computes the eigenvalues exactly and takes less number of computations compared with the Bisection method which is used in the existing method. The symmetric matrix is applicable to various Computer Engineering applications like Image processing, Pattern Recognition, Finger print Recognition and so on.

References:

[1] L. Wang et. al , “Global stability for monotone tridiagonal systems with negative feedback”, *IEEE Conference on decision and control*, pp. 2008,pp. 4091-4096.

[2] X. H. Sun et al., “Efficient tridiagonal solvers on multicomputer”, *IEEE Transactions on Computers*, Vol. 41, no. 3, 1992, pp. 286-296.

[3] Y. Juang and W. F. McColl,” A two-way BSP algorithm for tridiagonal systems”, *Future Generation Computer Systems*, Vol.13, Issues 4-5, 1998, pp. 337-347

[4] M. K. Jain, S. R. K. Iyengar and R. K. Jain, “Numerical Methods for Scientific and Engineering Computation”, Wiley Eastern Limited, 1993.

[5] V. S. Pusadkar and S. K. Katti, “A New Computational Technique to identify real eigenvalues of the system matrix A”, via Gerschgorin theorem, *Journal of Institution of Engineers (India)*, Vol. 78, Dec. 1997, pp. 121-123.

[6] Y.V. Hote,”New Approach of Kharitonov and Gerschgorin theorem in Control Systems”, Ph. D thesis, Delhi University, 2009.

[7] Y. V. Hote D. Roy Choudhury, and J. R. P. Gupta, “Gerschgorin theorem and its applications in control system problems”, *IEEE International Conference on Industrial Technology*, 2006, pp. 2438-2443.