SOLUTION OF MULTI-OBJECTIVE MATHEMATICAL PROGRAMMING PROBLEMS IN FUZZY APPROACH

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ABSTRACT

Recent developments in multi objective programming by Geoffrion, Mond and Wolfe [3, 8, 13] show interesting results with convex functions and related scalar objective programs. In this paper we compare the solution of multi objective linear programming problem with this solution obtained in Zimmermann's method. Zimmermann used membership function to solve the multi-objective linear programming problems. We have used α -cut to solve the multi objective linear programming problems.

Keywords: Multi-objective mathematical programming problems, fuzzy objective, α-cut, fuzzy triangular numbers.

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INTRODUCTION

In a multi-objective programming problem applied to real life model the data can rarely be determined exactly with certainty and precision. We may consider the intervals of real numbers and be sure that the data fluctuates in these intervals. If data fluctuation is not considered the resulting solution for the programming problem might be different from the optimal solution. It is clear that precision in decision making is very important and any error may give rise to high expenses in application. That is why in such problems, the data is considered in the form of fuzzy numbers with linear membership function [1, 6, 14, 15].

Zimmermann[15, 16] first applied fuzzy programming to multi-objective linear programming problems by using the concept given by Bellman and Zadeh [7,14]. In the last two decades several fuzzy programming techniques have been developed by various researchers [9, 10, 11, 12]. In this area more than 600 papers have been published all over the world. Lee and Hwang [4] have given a detailed survey on Fuzzy Mathematical Programming. It has been applied to many disciplines such as: advertising, assignment, blending, blood control, budgeting, computer selection, diet selection, disease control, drilling, employment, engg. design, finance, income tax, information services, investment location, maintenance, managerial decision making, manpower marketing, materials handling, media planning, medicare, metal cutting, networks, perishable products, personnel assignment, pollution, production, promotions, project selection, portfolio analysis, purchasing, quality control, racial balance, reliability, replacement, research and development, safety, salaries, sales, security scheduling, stock control, tendering, timetabling, transportation, visitors quota, and water resources management.

Zimmermann [15, 16] first classified fuzzy mathematical programming (FMP) method into two different models namely symmetric and non-symmetric models.

Leung [5] classified (FMP) into the following four categories:

i) a precise objective and fuzzy constraints,

ii) a fuzzy objective and precise constraints,

iii) a fuzzy objective and fuzzy constraints

iv) Robust programming (one of the possibilistic programming)

We present here a fuzzy programming approach to some crisp multi-objective decision making (MODM) problems.

A mathematical model for a MODM problem can be stated as:

Find $X = (x_1 \ x_2 \ \ldots.. \ x_n)^T$

so as to

maximize (minimize) $[\, f_1(X), f_2(X), \ldots.., f_k(X)], \ k = 1, 2, \ldots., K$ (1)

subject to

$$g_i(X) \quad (\le, =, \ge) \quad b_i, \quad i = 1, 2, \ldots\ldots.. m \tag{2}$$

$$x_j \ge 0, \ j = 1, 2, \ldots.., n \tag{3}$$

where $f_j(X)$, $j \in J$ are the benefit maximization objectives, $f_i(X)$, $i \in I$ are the cost minimization objectives and $I \cup J = 1, 2, \ldots., K$. It is noted that all functions $f_k(X)$, $k = 1,2,\ldots K$ and $g_i(X)$, $i = 1, 2, \ldots.., m$ may be linear or non-linear. If all the objective functions are of maximization type, then the problem is known as Vector Maximization Problem. If all are of minimization type, then it is known as Vector Minimization Problem.

MULTI-OBJECTIVE LINEAR PROGRAMMING MODEL (VECTOR MAXIMUM PROBLEM)

A mathematical model can be stated as :

Find $X = (x_1 \ x_2 \ \ldots.. \ x_n)^T$

so as to

maximize $$Z_k(X) = \sum_{j=1}^{n} C_j^k x_j, \ k = 1, 2, \ldots.., K \tag{4}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \ (\leq, =, \geq) \quad b i, \quad i = 1, 2, \ldots\ldots m \tag{5}$$

$$x_j \geq 0, \ j = 1, 2, \ldots.. n. \tag{6}$$

It is assumed that the objective functions are crisp but the objectives are conflicting in nature. It is also assumed that the problem is feasible and there exits an optimal compromise solution. We apply fuzzy programming approach to find an optimal compromise solution. The steps of the method are as follows:

ZIMMERMAN'S METHOD

Step-1: Solve the multi-objective linear programming problem as a single objective linear programming problem by using any linear programming algorithm, considering only one of the objectives at a time and ignoring all others. Repeat the process K times for K different objective functions.

Let $X^{(1)}$, $X^{(2)}$, ……., $X^{(K)}$ be the ideal solutions for the respective objective functions.

Step-2: Using all the above ideal solutions in Step 1, construct a pay-off matrix of size K by K. Then from the pay-off matrix estimate the lower bound (L_k) and the upper bound (U_k) for the kth objective function Z_k as :

$$L_k \leq Z_k \leq U_k, \qquad k = 1, 2 \ldots.. , K$$

Step – 3: Define a fuzzy linear membership function (μZ_k (X)) [2, 14, 15] for the kth objective function Zk, k = 1, 2 ….. , K, [5]

$$\mu Z_k(X) = \begin{cases} 0 & \text{if } Z_k \leq L_k \\ 1 - \dfrac{(U_k - Z_k)}{U_k - L_k} & \text{if } L_k \leq Z_k \leq U_k \\ 1 & \text{if } Z_k \geq U_k \end{cases} \tag{7}$$

Step 4: Use the above membership functions to formulate a crisp model by introducing an augmented variable λ.

$$\text{Maximize:} \qquad 1 - \frac{(U_k - Z_k)}{U_k - L_k}, \qquad k = 1, 2 \ldots.. , K \tag{8}$$

$$\text{Minimize:} \qquad \frac{(U_k - Z_k)}{U_k - L_k} \qquad k = 1, 2 \ldots.. , K \tag{9}$$

subject to (5) and (6)

can be further simplified as:

$$\text{Minimize: } \lambda \tag{10}$$

subject to

$$\sum_{j=1}^{n} C_j^k x_i + (U_k - L_k)\lambda \geq U_k \text{, if } k = 1, 2 \text{ ….. }, K \tag{11}$$

$$\sum_{j=1}^{n} a_{ij} x_j (\leq, =, \geq) \text{ bi, } i = 1, 2 \text{ ….. }, m \tag{12}$$

$$\lambda \geq 0,\ xj \geq 0,\ j = 1, 2, \text{ ……}, n \tag{13}$$

Step-5: Solve the crisp model (as stated in equation (10). (13)) by an LP algorithm and find the optimal compromise solution X*. Evaluate all the objective functions at the optimal compromise solution X*.

NUMERICAL EXAMPLE

A numerical example with two objective functions, three constraints and two variables is considered to illustrate the solution procedure.

Find $X = (x_1, x_2)^T$, so as to

$$\text{maximize:} \quad \begin{cases} Z_1 = x_1 + 3x_2 \\ Z_2 = 3x_1 + x_2 \end{cases}$$

subject to

$x_1 + x_2 \leq 10$

$x_1 + 2x_2 \leq 11$

$x_1, x_2 \geq 16$

For the first objective function the ideal solution is obtained as:

$$X^{(1)} = \begin{pmatrix} x_1^{(1)} = 6 \\ x_2^{(1)} = 2.5 \end{pmatrix}$$

and $Z_1 = 13.5$

For the second objective function the ideal solution is obtained as:

$$X^{(2)} = \begin{pmatrix} x_1^{(2)} = 10 \\ x_2^{(2)} = 0 \end{pmatrix}$$

and $Z_2 = 30.0$

A pay-off matrix is formulated as:

	Z_1	Z_2
$X^{(1)}$	13.5	20.5
$X^{(2)}$	10.0	30.0

From the pay-off matrix, lower bound and the upper bound are estimated as:

$$10 \leq Z_1 \leq 13.5$$

$$20.5 \leq Z_2 \leq 30$$

Using the membership functions as defined in equation (7) and introducing and augmented variable a crisp model is formulated as:

Minimize: λ

subject to

$$x_1 + 3x_2 + 3.5\lambda \geq 13.5$$

$$3x_1 + x_2 + 9.5\lambda \geq 30$$

$$x_1 + x_2 \leq 10$$

$$x_1 + 2x_2 \leq 11$$

$$x_1 + 4x_2 \leq 16$$

$$\lambda, x_1, x_2 \geq 0$$

Finally, the crisp model is solved to find the optimal compromise solution as:

$$X^* = \begin{pmatrix} x_1 = 8.4629 \\ x_2 = 1.2685 \\ \lambda = 0.3518 \end{pmatrix}$$

The values of the objective function at X* is obtained as:

$$Z_1^* = 12.269, \; Z_2^* = 26.657$$

If we consider a vector minimum LP problem, then the same fuzzy programming method can be used. However, one should redefine the membership functions. Other steps remain unchanged.

MODIFIED METHOD (MAIN RESULT)

A mathematical model of multi objective linear programming problem in real plane is,

Find $\mathrm{X} = (x_1, .., x_n)^T$

so as to

$$\text{maximize / minimize : } Z_k(x) = \sum_{j=1}^{n} C_j{}^k x_j, k = 1, 2, .., k \tag{14}$$

$$\text{subject to : } \sum a_{ij} x_j (\leq = \geq) b_i, i = 1, 2, .., m \;\; x_j \geq 0, j = 1, 2, .., n \tag{15}$$

In our result we assume the variable set $X = (x_1 \;\; x_2 \;...x_n)^T$ and right hand side vector $'b_i'$ of constraints are in fuzzy domain. Let x_i's and b_i's be fuzzy triangular numbers and the matrix A is in crisp form whose elements are real numbers.

ALGORITHM

Step-1: Define the membership function corresponding to X as

$$\mu_{x_i}(X) = \begin{cases} 0 \text{ if } x_i < \underline{xi} \\ \dfrac{\overline{x_i} - x_i}{\overline{\overline{x_i}} - \underline{x_i}} \text{ if } \underline{x_i} < x_i < \overline{x_i} \\ 1 \text{ if } x_i > \overline{x}i \end{cases} \tag{16}$$

Step-2: Use α - cut to make the fuzzy system to crisp and use general method to solve the system.

$$\begin{aligned} &\text{For any } \alpha \in [0,1], \\ &\frac{\overline{x_i} - x_i}{\overline{\overline{x_i}} - \underline{x_i}} = \alpha \\ &\text{if } x_i = (1-\alpha)\overline{x_i} + \alpha \underline{x_i} \end{aligned} \tag{17}$$

Step-3: Using this α - cut we change the multi objective linear programming problem as:

$$\text{Maximize/Minimize} \quad Z_k(x) = \sum c_j{}^k \left((1-\alpha)\overline{x}_j + \alpha \underline{x}_j \right), 0 \leq \alpha \leq 1, k = 1, 2, ..., m \tag{18}$$

$$\text{subject to} \quad \sum_{j=1}^{n} a_{ij} \left((1-\alpha)\overline{x}_j + \alpha \underline{x}_j \right) \leq (\geq, =) (1-\alpha)\overline{b_i} + \alpha \underline{b}_i, i = 1, 2........, m \tag{19}$$

$$\text{and} \quad x_j \geq 0$$

Step-4: By putting α =0 and α =1 obtain the lower bound (x_i) and upper band($\overline{x}_i$)of the optimal solution of that MOLPP (4).

Step-5: Obtain the optimal solution of the MOLPP by average of the lower and upper bound of the solution. Also by taking the average of the point we get the point where optimal solution will exist. Repeat this process k-times for k different objective functions.

NUMERICAL EXAMPLE

A numerical example with two objective functions, two variables and three constraints is considered.

Find $X = (\tilde{x}_1, \tilde{x}_2)^T$

so at to

maximize $\begin{cases} z_1 = \tilde{x}_1 + 3\tilde{x}_2 \\ z_2 = 3\tilde{x}_1 + \tilde{x}_2 \end{cases}$

subject to: $\tilde{x}_1 + \tilde{x}_2 \le \tilde{1}0$

$$\tilde{x}_4 + 2\tilde{x}_2 \le \tilde{1}1$$

$$\tilde{x}_4 + 4\tilde{x}_2 \le \tilde{1}6$$

$$\tilde{x}_1, \tilde{x}_2 \ge 0$$

We use fuzzy triangular numbers:

$$\tilde{1}0 = [8,10,12]$$

$$\tilde{1}1 = [10,11,13]$$

$$\tilde{1}6 = [14,16,18]$$

Using the equation, the MOLPP becomes

maximize $\begin{cases} z_1 = (1-\alpha)\tilde{x}_1 + \alpha\underline{x_1} + (1-\alpha)3\overline{x}_2 + \alpha 3\underline{x}_2 \\ z_2 = (1-\alpha)3\tilde{x}_1 + \alpha\underline{x_1} + (1-\alpha)\overline{x}_2 + \alpha\underline{x}_2 \end{cases}$

subject to: $(1-\alpha)\tilde{x}_1 + \alpha\underline{x_1} + (1-\alpha)\overline{x}_2 + \alpha\underline{x}_2 \le (1-\alpha).12 + 8\alpha$

$$(1-\alpha)\tilde{x}_1 + \alpha\underline{x_1} + (1-\alpha)\overline{2x}_2 + \alpha.2\underline{x}_2 \le (1-\alpha)13 + 10\alpha$$

$$(1-\alpha)\tilde{x}_1 + \alpha.\underline{x_1} + (1-\alpha)\overline{4x}_2 + \alpha.4\underline{x}_2 \le (1-\alpha)18 + 14\alpha$$

$$\overline{x}_1, \underline{x}_1, \overline{x}_2, \underline{x}_2 \ge 0$$

We solve by taking the first objective function.

maximize $Z_1 = (1-\alpha)\overline{x}_1 + \alpha\underline{x}_1 + (1-\alpha)3\overline{x}_2 + \alpha 3\underline{x}_2$

subject to : $(1-\alpha)\overline{x}_1 + \alpha\underline{x}_1 + (1-\alpha)\overline{x}_2 + \alpha x_2 \le 12 - 4\alpha$

$$(1-\alpha)\overline{x}_1 + \alpha\,\underline{x}_1 + (1-\alpha)2\overline{x}_2 + \alpha.2\underline{x}_2 \le 13 - 3\alpha$$

$$(1-\alpha)\overline{x}_1 + \alpha\,\underline{x}_1 + (1-\alpha)4\overline{x}_2 + \alpha.4\underline{x}_2 \le 18 - 4\alpha$$

$$\underline{x}_1, \overline{x}_1, \underline{x}_2, \overline{x}_2 \geq 0$$

For $\alpha = 0$ the MOLPP becomes,

Maximize $\quad Z_1 = \overline{x}_1 + 3\overline{x}_2$

subject to : $\quad \overline{x}_1 + \overline{x}_2 \leq 12$

$$\overline{x}_1 + 2\overline{x}_2 \leq 13$$

$$\overline{x}_1 + 4\overline{x}_2 \leq 18$$

$$\overline{x}_1, \overline{x}_2 \geq 0$$

and the optimal solution is at $(10, 2)$ with $Z_1 = 16$

For $\alpha = 1$ the MOLPP becomes

Maximize $\quad Z_1 = \underline{x}_1 + 3\underline{x}_2$

subject to: $\quad \underline{x}_1 + \underline{x}_2 \leq 8$

$$\underline{x}_1 + 2\underline{x}_2 \leq 10$$

$$\underline{x}_1 + 4\underline{x}_2 \leq 14$$

$$\underline{x}_1, \underline{x}_2 \geq 0$$

Optimal solution is at (6, 2) with $Z_1 = 12$

Hence the optimal solution of MOLPP occurs at $\left(\frac{10+6}{2}, \frac{2+2}{2}\right) = (8, 2)$ with optimal objective an value $\frac{16+12}{2} = 14$.

For the second objective function the MOLPP is

Maximize $\quad Z_2 = (1-\alpha)3\overline{x}_1 + \alpha.3\underline{x}_1 + (1-\alpha)\overline{x}_2 + \alpha\underline{x}_2$

subject to $\quad (1-\alpha)\overline{x}_1 + \alpha\underline{x}_1 + (1-\alpha)\overline{x}_2 + \alpha\underline{x}_2 \leq 12 - 4\alpha$

$$(1-\alpha)\overline{x}_1 + \alpha\underline{x}_1 + (1-\alpha)2\overline{x}_2 + \alpha.\underline{x}_2 \leq 13 - 3\alpha$$

$$(1-\alpha)\overline{x}_1 + \alpha\underline{x}_1 + (1-\alpha)4\overline{x}_2 + \alpha 4\underline{x}_2 \leq 18 - 4\alpha$$

$$\underline{x}_1, \overline{x}_1, \underline{x}_2, \overline{x}_2 \geq 0$$

For $\alpha = 0$ the MOLPP becomes,

Maximize $\quad Z_1 = 3\overline{x}_1 + \overline{x}_2$

subject to: $\quad \overline{x}_1 + \overline{x}_2 \leq 12$

$$\overline{x}_1 + 2\overline{x}_2 \leq 13$$

$$\overline{x}_1 + 4\overline{x}_2 \leq 18$$

$$\overline{x}_1, \overline{x}_2 \geq 0$$

The optimal solution is occurs at (10, 2) with Z_2 = 32.

For $\alpha = 1$ the MOLPP becomes

Maximize $\quad Z_1 = 3\underline{x}_1 + \underline{x}_2$

subject to : $\quad \underline{x}_1 + \underline{x}_2 \leq 8$

$$\underline{x}_1 + 2\underline{x}_2 \leq 10$$

$$\underline{x}_1 + 4\underline{x}_2 \leq 14$$

$$\underline{x}_1, \underline{x}_2 \geq 0$$

The optimal solution is at (8, 0) with Z_2 = 24.

Hence the optimal solution of MOLPP occurs at $\left(\frac{10+8}{2}, \frac{2+0}{2}\right) = (9,1)$ with optimal value $\frac{32+24}{2} = 28.$

CONCLUSION

Zimmermann's method is an eye opener and it guarantees stable and a crisp fixed solution to multi-objective mathematical programming problems. The modified method presented by the authors in this paper using α -cut and fuzzy triangular numbers extends the solution to an interval on the real line and hence generalizes Zimmermann's method. We conclude that a number of fuzzy optimal solutions are possible on the considered interval.

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