Comparison of Optimal Homotopy Asymptotic Method with Homotopy Perturbation Method of Twelfth Order Boundary Value Problems

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Abstract

In this work, we consider special problem consisting of twelfth order two-point boundary value by using the Optimal Homotopy Asymptotic Method and Homotopy Perturbation Method. Now, we discuss the comparison in between Optimal Homotopy Asymptotic Method and Homotopy Perturbation Method. These proposed methods have been thoroughly tested on problems of all kinds and shows very accurate results. A numerical example is present and approximate is compared with exact solution and the error is compared with Optimal Homotopy Asymptotic Method and Homotopy Perturbation Method to assess the efficiency of the Optimal Homotopy Asymptotic Method at 12^{th} order Boundary values problems.

Keywords

Twelfth order boundary value problems, Approximate analytical solution, homotopy perturbation Method, optimal homotopy Asymptotic method, Ordinary Differential Equations, Error Estimates.

1. Introduction

In literature different techniques are available for the numerical solution of twelfth order boundary value problems. The motivation of this problem is to extend Optimal Homotopy Asymptotic Method to solve linear and non linear tenth order boundary value problems. We also compared the results obtained from these techniques with the available exact solution in different literatures. Some properties of solutions of a given differential equation may be determined without finding their exact form in especially in nonlinear behavior. If as self-contained formula for the solution is not available, the solution may be numerically approximated using computers. To overcome these difficulties, a modified form of the perturbation method called homotopy perturbation method.HPM has since then been effectively utilized in obtaining approximate analytical solutions to many linear and nonlinear problems arising in engineering and science, such as nonlinear oscillators with discontinuities [3], nonlinear Volterra –Fredholm integral equations [11], Twelfth order differential equations have several important applications in engineering. Solution of linear and nonlinear boundary value problems of twelfth-order was implemented by Wazwaz using Adomian decomposition method. Chandrasekhar [9] showed that when an infinite horizontal layer of fluid is put into rotation and simultaneously subjected to heat from below and a uniform magnetic field across the fluid in the same direction as gravity, instability will occur. Several researchers developed numerical techniques for solving twelfth order differential equations. The Adomian Decomposition Method [1, 4], the Differential Transform Method [15], the Variational Iteration Method, the successive iteration, the splines [5, 6], the Homotopy Perturbation Method [7], the Homotopy Analysis Method etc Recently Vasile Marinca et al. [10,12,14] introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. OHAM is straight forward, reliable and it does not need to look for h curves like HPM. Moreover, this method provides a convenient way to control the convergence of the series solution. Most recently, Javed Ali et al. used OHAM for the solutions of multi-point boundary value problems. The results of OHAM presented in this work are compared with those of exact solution HPM.

2. Basic Idea of Homotopy-Perturbation Method

To clarify the basic ideas of the homotopy perturbation method [13], we consider the following nonlinear differential equation

$$A(u) - f(r) = 0, \quad r \in \Omega \tag{1}$$

with boundary conditions

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma \tag{2}$$

where A is a general differential operator, B is a boundary operator, u is a known analytical function, and Γ is the boundary of the domain Ω. The operator A can be divided into two parts L and N, where L is linear, while N is nonlinear. Therefore (1) can be rewritten as follows

$$L(u) + N(u) - f(r) = 0. \tag{3}$$

By the homotopy technique, we can construct a homotopy

$V(r, p): \Omega \times [0, 1] \to R$ which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1] \in \Omega \tag{4}$$

or

$$H(v, p) = L(v) - L(u_0) + p L(u_0) + p[N(v) - f(r)] = 0, \tag{5}$$

where $r \in \Gamma$ and $p \in [0, 1]$ is an embedding parameter, u_0 is an initial approximation of (1), which satisfies the boundary conditions. Obviously, from Equations (4) and (5) we will have:

$$H(v, 0) = L(v) - L(u_0) = 0, \tag{6}$$

$$H(v, 1) = A(v) - f(r) = 0, \tag{7}$$

and the changing process of p from zero to unity is just that of H(v, p) from $L(v)-L(u_0)$ to $A(v)-f(r)$. In topology, this is called deformation, $L(v) - L(u_0)$ and $A(v) - f(r)$ is called homotopic. The embedding parameter p is introduced much more naturally, unaffected by artificial factors. Furthermore, it can be considered as a small parameter for $0 < p \leq 1$. So it is very natural to assume that the solution of (4, 5) can be expressed as

$$v = v_0 + p v_1 + p^2 v_2 + \cdot \cdot \cdot .$$

Therefore, when p=1, the approximate solution of above equation can be readily obtained as follows:

$$u = \lim_{p \to 1} v_0 + v_1 + v_2 + \cdot \cdot \cdot .$$

The combination of the perturbation method and the homotopy method is called the HPM, which eliminates the drawbacks of the traditional perturbation methods while keeping all its advantages. The above series is convergent for most cases.
Using the transformation

$$y = y_1, \frac{dy}{dx} = y_2, \frac{d^2y}{dx^2} = y_3, \ldots\ldots\ldots\ldots, \frac{d^{2n-1}y}{dx^{2n-1}} = y_{2n} \tag{8}$$

Using the boundary conditions we can be written as a system of integral equations:

$y_1 = a + \int_0^x y_2(t)dt$, $y_2 = b + \int_0^x y_3(t)dt$, $y_3 = c + \int_0^x y_4(t)dt$ ………………….

$y_{2n} = d + \int_0^x f\{t, y_1(t), y_2(t), y_3(t), \ldots . y_{2n-1}(t)\}dt$

3. Optimal Homotopy Asymptotic Method Analysis

The general differential equation is considered as,

$$L\,(u(x)) + g(x) + N\,(u(x)), \quad B\,(u, \frac{du}{dn}) = 0 \tag{9}$$

where L is a linear operator, x denotes independent variable, u(x) is an unknown function, g(x) is a known function, N (u(x)) is a nonlinear operator and B is a boundary operator. A family of equations is constructed using (OHAM) which is given by

$$(1 - p)L\,(f\,(x, p) + g(x) = H\,(p)\,(L\,(f\,(x, p) + g(x) + N\,(f\,(x, p))$$

$$B\,(f\,(x, p), \partial f\,(x, p) / \partial x). \tag{10}$$

Where $p \in [0,1]$ is an embedding parameter, H (p) is a nonzero auxiliary function for $p \neq 0$ and $H(0) = 0$,f (x, p) is an unknown function, respectively. Obviously, when p=0 and p = 1, it holds

$$f\,(x,0) = u_0\,(x),\ f\,(x,1) = u_1\,(x), \tag{11}$$

respectively. Thus, as p increases from 0 to 1, the solution f (x, p) varies from u_0 (x) to the solution u_1 (x), where u_0 (x) is obtained from equation (10) for p=0:

$$L\,(u_0\,(x)) + g(x) = 0,\ B\,(u_0, d\,u_0 / d\,x) = 0 \tag{12}$$

Auxiliary function H (p) is chosen in the form

$$H\,p = pc_1 + p^2 c_2 + \ldots\ldots\ldots\ldots \tag{13}$$

where constants c_1, c_2 ,... can be determined latter by the method of least squares. Let us consider the solution of Eq. (10) in the form to get an approximate solution; we used Taylor's series to expand f(x, p, C_i) in p which becomes

$$f(x, p, c_i) = u_0\,(x) + \sum_{m=1}^{\infty} u_0(x, c_1, c_2, c_{3,} \ldots \ldots \ldots \ldots c_m)p^m \tag{14}$$

Now substituting Eq. (14) into Eq. (10) and equating the coefficients of like powers of p,, we obtain the following linear equations. Zeroth order problem is given by Eq. (12) and the first order problem is given by

$$L\,(u_1(x)) + g(x) = C_1 N_0\,(u_0(x)), \quad B\,(u_1, \frac{du_1}{dn}) = 0 \tag{15}$$

The mth -order problem is given by:

$$L\,(u_m(x)) - L\,(u_{m-1}(x)) = C_m N_0\,(u_0(x) +$$

$$\sum_{i=1}^{m-1} c_i [L\left(u_{m-1}(x)\right) + N_{m-1}\left(u_0(x), u_1(x), u_2(x), \ldots \ldots, u_{m-1}(x)\right)] \tag{16}$$

Here m varies 2, 3, 4………..B $(u_m, \frac{du_m}{dn}) = 0$

where Nm (u_0(x), u_1(x),…, u_m(x)) is the coefficient of p^m in the expansion of N(n(x, p)) about the embedding parameter p.

$$\mathrm{Nf}(x, p, c_i) = N_0 u_0\left((x)\right) + \sum_{m=1}^{\infty} N_m\left(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_m\right)p^m$$

It has been observed that the convergence of the series (14) depends upon the auxiliary constants C_1, C_2,……….. If it is convergent at p = 1, one has

$$f(x, c_i) = u_0\ (x) + \sum_{m=1}^{\infty} u_m(c_1, c_2, c_{3,} \dots \dots \dots \dots c_m)$$

The result of the mth-order approximations are given

$$u\left(c_1, c_2, c_{3,} \dots \dots \dots \dots c_m\right) = u_0\left((x)\right) + \sum_{i=1}^{m} u_i\left(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_i\right) \qquad (17)$$

Substituting Eq. (17) into Eq. (9), it results the following residual:

$$R\left(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_m\right) = L(u\left(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_m\right)) + g(x) + Nu(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_m)$$

If R = 0, then u% will represent the exact solution. Generally it doesn't happen, especially in nonlinear problems. To have a minimum value of |R| for the values, C_i here i = 1, 2, 3… various methods for instance the weighted residual method or variational methods etc. can be used. By using the Least squares method we obtain the following equation

$$\int_a^b R\ \left(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_m\right)\frac{\partial R}{\partial c_i}\ dx = 0 \qquad (18)$$

while the Galerkin's method gives

$$\int_a^b R\ \left(x, c_1, c_2, c_{3,} \dots \dots \dots \dots c_m\right)\frac{\partial R}{\partial c_i}\ dx = 0 \qquad (19)$$

where a and b are in the domain of the problem. For the given value of all constants we can easily determine the approximate solution of order m.

4. NUMERICAL EXAMPLES

Example 1: Consider the following linear twelfth order boundary value problem

$y^{12}(x) = -x\ y(x) - x^3e^x - 23xe^x - 120e^x$

with following conditions:

$${}_0 = 0, \quad {}^{(1)}_0 = 1\ , \quad {}^{(2)}_0 = 0\ \ , \quad {}^{(3)}_0 = -3, \quad {}^{(4)}_0 = -8, \quad {}^{(5)}_0 = -15$$

$${}_1 = 0, \quad {}^{(1)}_1 = -e, \quad {}^{(2)}_1 = -4e, \quad {}^{(3)}_1 = -9e, \quad {}^{(4)}_1 = -16e, \quad {}^{(5)}_1 = -25e \qquad (20)$$

Exact solution is $y\ (x) = x\ (1 - x)\ e^x$

Using the transformation (8) we can rewrite the twelve-order boundary value problem (20) as the system of integral equations:

$$y_1 = 0 + \int_0^x y_2(t)dt$$

$$y_2 = 1 + \int_0^x y_3(t)dt$$

$$y_3 = 0 + \int_0^x y_4(t)dt$$

$$y_4 = -3 + \int_0^x y_5(t)dt$$

$$y_5 = -8 + \int_0^x y_6(t)dt$$

$$y_6 = -15 + \int_0^x y_7(t)dt$$

$$y_7 = a + \int_0^x y_8(t)dt$$

$$y_8 = b + \int_0^x y_9(t)dt$$

$$y_9 = c + \int_0^x y_{10}(t)dt$$

$$y_{10} = d + \int_0^x y_{11}(t)dt$$

$$y_{11} = e + \int_0^x y_{12}(t)dt$$

$$y_{12} = f + \int_0^x \{-xy_1(t) - 120e^t - 23\,te^t\text{-}t^3e^t\}dt$$

Comparing the coefficients of like powers of p, we have:

Coefficients of p^0:

$y_{10} = 0$	$y_{20} = 1$	$y_{30} = 0$	$y_{40} = -3$	$y_{50} = -8$	$y_{60} = -15$
$y_{70} = a$	$y_{80} = b$	$y_{90} = c$	$y_{100} = d$	$y_{110} = e$	$y_{120} = f$

Coefficients of p^1:

$y_{11} = x$	$y_{21} = 1$	$y_{31} = -3x$	$y_{41} = -3-8x$
$y_{51} = -8 -15x$	$y_{61} = -15+ax$	$Y_{71} = a+ b\,x$	$Y_{81} = b +c\,x$
$y_{91}= c+ d\,x$	$y_{101}= d +ex$	$Y_{111}= e +f\,x$	$Y_{121}= f-e^x\,(x^3- 3x^2 + 91+29x)$

Coefficients of p^2:

$y_{12}= x$	$y_{22}= 1- 3\frac{x^2}{2}$	$y_{32}=-3x-x^2$
$y_{42}= -3-8x- \frac{15}{2}\,x^2$	$y_{52}=-8-15x- \frac{a}{2}x^2$	$y_{62}= - 15+ a\,x+ \frac{b}{2}x^2$
$y_{72}= a+ b\,x+ \frac{c}{2}x^2$	$y_{82}= b+ c\,x+ \frac{d}{2}x^2$	$y_{92}= c+ d\,x+ \frac{e}{2}x^2$
$Y_{102}= d+ e\,x+ \frac{f}{2}x^2$		
$Y_{112}= e + 50 +(91 +f)x -e^x\,(50+41x-6x^2 +x^3)$		
$Y_{122}= f +91- \frac{x^3}{3} -e^x\,(50+29x - 3x^2 +x^3)$		

Coefficients of p^3:

$y_{13}= x- \frac{x^3}{2}$	$y_{23}= 1- 3\frac{x^2}{2} - \frac{4}{3}x^2$	$y_{33}=-3x-2x^2- \frac{5}{2}x^3$
$y_{43}= -3-8x- \frac{15}{2}\,x^2 +\frac{a}{6}x^3$	$y_{53}=-8-15x- \frac{a}{2}x^2 +\frac{b}{6}x^3$	$y_{63}= - 15+ a\,x+ \frac{b}{2}x^2 +\frac{c}{6}x^3$
$y_{73}= a+ b\,x+ \frac{c}{2}x^2 +\frac{d}{6}x^3$	$y_{83}= b+ c\,x+ \frac{d}{2}x^2 +\frac{f}{6}x^3$	$y_{93}= c+ d\,x+ \frac{e}{2}x^2+\frac{f}{6}x^3$
$Y_{103}= -9 +d +(50 +e)x + 0.5(91 +f)x^2 -e^x\,(-9+59x-9x^2 +x^3)$		
$Y_{113}= e + 50 +(91 +f)x -e^x\,(50+41x-6x^2 +x^3) - \frac{x^4}{12}$		

$Y_{123}= f +91- \frac{x^3}{3} - e^x (91+29x - 3x^2 +x^3)$

Coefficients of p^4:

$y_{14}= x- \frac{x^3}{2} - \frac{x^4}{3}$	$y_{24}= 1- 3\frac{x^2}{2} - \frac{4}{3}x^2- \frac{5x^4}{8}$	$y_{34}=-3x-2x^2- \frac{5}{2}x^3 + \frac{ax^4}{24}$
$y_{44}= -3-8x- \frac{15}{2} x^2 +\frac{a}{6}x^3+ \frac{bx^4}{24}$		
$y_{54}=-8-15x- \frac{a}{2}x^2 +\frac{b}{6}x^3+ \frac{cx^4}{24}$		
$y_{64}= - 15+ a x+ \frac{b}{2}x^2 +\frac{c}{6}x^3+ \frac{dx^4}{24}$		
$y_{74}= a+ b x+ \frac{c}{2}x^2 +\frac{d}{6}x^3+ \frac{f x^4}{24}$		
$y_{84}= b+ c x+ \frac{d}{2}x^2 +\frac{e}{6}x^3+ \frac{f x^4}{24}$		
$y_{94}= -92 +c +(-9 +d)x + 0.5(50 +f)x^2 - e^x (-92+83x-12x^2 +x^3)+(91 +f)\frac{1}{6}x^3$		
$y_{104}= -9 +d +(50 +e)x + 0.5(91 +f)x^2 - e^x (-9+59x-9x^2 +x^3) - \frac{x^5}{60}$		
$y_{114}= e + 50 +(91 +f)x - e^x (50+41x-6x^2 +x^3) - \frac{x^4}{12}$		
$y_{124}= f +91- \frac{x^3}{3} - e^x (91+29x - 3x^2 +x^3) + \frac{x^5}{60}$		

Above procedure is proceeding up to coefficient of p^{12}.Now, adding above all coefficient of p^0 to p^{12}, and then we obtain:

$$y^{12}(x) = x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{8}x^5 + \frac{a}{720}x^6 + \frac{b}{5040}x^7 + \frac{c}{40320}x^8 + \frac{d}{362880}x^9 + \frac{e}{3628800}x^{10} + \frac{f}{39916800}x^{11} - \frac{1}{3991680}x^{12} - \frac{1}{43545600}x^{13} - \frac{83}{43589145600}x^{14} - \frac{1}{6706022400}x^{15} + \ldots\ldots\ldots\ldots$$

The coefficients a, b, c, d, e, f can be obtained using the boundary conditions at x =1,

a = 23.9999985, b =35.000057, c = 47.998961, d =63.0108031, e =79.9359481, f= 99.17376631.

The series solution can, thus, be written as

$$y^{12}(x) = x - \frac{1}{2}x^3 - \frac{1}{3}x^4 - \frac{1}{8}x^5 + 0.0333333\, x^6 - 0.00694446\, x^7 - 0.00119045\, x^8 - 0.000173641x^9 - 0.0000220282\, x^{10} +.00000248451x^{11} - \frac{1}{3991680}x^{12} - \frac{1}{43545600}x^{13} - \frac{83}{43589145600}x^{14} - \frac{1}{6706022400}x^{15} + \ldots\ldots\ldots\ldots$$

x	Analytical Solution	Numerical Solution	Errors
0.0	1.000000000	0.0000000000	0.00000
0.1	0.0994653826	0.0994653826	3.00×10^{-11}
0.2	0.1954244413	0.1954244413	0.00000
0.3	0.2834703497	0.2834703496	-1.00×10^{-10}
0.4	0.3580379275	0.3580379277	2.00×10^{-10}
0.5	0.4121803178	0.4121803189	1.10×10^{-9}
0.6	0.4373085120	0.4373085164	4.40×10^{-9}
0.7	0.4228880685	0.4228880820	1.35×10^{-8}
0.8	0.3560865485	0.3560865853	3.68×10^{-8}
0.9	0.2213642800	0.2213643701	9.01×10^{-8}
1.0	0.0000000000	0.0000002027	2.02700×10^{-7}

Table: [1.1]

Example 2: Consider the following linear twelfth order boundary value problem

$$y^{12}(x) = -x\, y(x)-x^3e^x -23xe^x -120e^x$$

with following conditions:

$$y_0 = 0,\ y_0^{(1)} = 1\ , y_0^{(2)} = 0\ ,\ y_0^{(3)} = -3,\quad y_0^{(4)} = -8,\quad y_0^{(5)} = -15$$

$$y_1 = 0,\ y_1^{(1)} = -\mathrm{e},\quad y_1^{(2)} = -4\mathrm{e},\quad y_1^{(3)} = -9\mathrm{e},\quad y_1^{(4)} = -16\mathrm{e},\ y_1^{(5)} = -25\mathrm{e}$$

Exact solution is $y(x) = x(1- x)e^x$

We construct the following zeroth and first-order problems.

$$y_0^{12}(x) = -x\, y(x)-x^3e^x -23xe^x -120e^x$$

with following conditions:

$$y_0(0) = 0,\ y_0^{(1)}(0) = 1\ , y_0^{(2)}(0) = 0\ ,\ y_0^{(3)}(0) = -3,\quad y_0^{(4)}(0) = -8,\quad y_0^{(5)}(0) = -15$$

$$y_0(1) = 0,\ y_0^{(1)}(1) = -\mathrm{e},\quad y_0^{(2)}(1) = -4\mathrm{e},\quad y_0^{(3)}(1) = -9\mathrm{e},\ y_0^{(4)}(1) = -16\mathrm{e},\ y_0^{(5)}(1) = -25\mathrm{e}$$

First-Order Problem:

$$y_1^{12}(x) = (1+ C_1)(120 + 23x + x^3)e^x + C_1x\, y_0(x) + (1 + C_1)\, y_0^{12}(x)$$

With same above boundaries' conditions Solutions to these problems are given by Equations. (20) and (21) respectively

$$y_0(x) = \frac{1}{120}(-280800 +280800\, e^x -221760x -58920\, e^xx-85800x^2+4320e^xx^2 -21600x^3-120\, e^xx^3-3960x^4$$

$$-560x^5+197720040x^6+72737161ex^6- 836896800x +307877125ex^7+ 1451896600ex^8$$

$$-1282301040x^9+ 471732190ex^9+ 5741817x^{10}- 211229665ex^{10} -103986080x^{11}$$

$$+38254341ex^{11}) \quad (20)$$

$$y_1 (x, C_1) = C_1 (216060-216060e^x+ (176352\ 39708\ e^x)\ x+ (71045-2723\ e^x)\ x^2+ (18795+ 84\ e^x)\ x^3$$
$$+ (3663 - e^x)\ x^4+ 558.833x^5+ 69.1417x^6+7.07738x^7+ 0.603671x^8+0.0425263x^9+$$

$$+0.00237265x^{10}+0.0000892391x^{11}-3.75782\times 10^{-7}x^{13}- 4.23959 \times 10^{-8}x^{14} - 3.2806310^{-9}x^{15}$$

$$-2.06473\ 10^{-10}x^{16}-1.11334 \times 10^{-11}x^{17} -5.24805 \times 10^{-12}x^{18}- 2.17518 \times 10^{-14}x^{19}$$

$$-7.89181 \times 10^{-16}x^{20} -2.46619\times 10^{-17}x^{21} - 6.40591 \times 10^{-19}x^{22} -1.27589 \times 10^{-20}x^{23}$$

$$-1.55619\ 10^{-22}x^{24}) \quad (21)$$

Considering the OHAM first-order solution,

$$Y_{app} (x, C_1) = y_0(x) + y_1 (x, C_1) \quad (22)$$

and using Eq.(18) with a = 0.5 and b = 1, we get $C_1 = -0.00260417$. Using this value the first-order solution (22) is well-determined.

x	Analytical Solution	Numerical Solution	Errors
0.0	1.000000000	0.000000000	0.00000
0.1	0.099465383	0.099465383	-7.5065×10^{-14}
0.2	0.195424441	0.195424441	-2.7686×10^{-12}
0.3	0.283470350	0.283470350	-1.7271×10^{-11}
0.4	0.358037927	0.358037927	-5.0232×10^{-11}
0.5	0.412180318	0.412180318	-9.3401×10^{-11}
0.6	0.437308512	0.437308512	-1.2791×10^{-10}
0.7	0.422888068	0.422888069	-1.3917×10^{-10}
0.8	0.356086548	0.356086549	-1.2278×10^{-10}
0.9	0.221364280	0.221364280	-7.4997×10^{-11}
1.0	0.000000000	-0.000000000	1.9454×10^{-11}

Table: [1. 2]

5. Conclusion

In this paper, the Comparison of the results obtained by the Homotopy perturbation method and optimal homotopy asymptotic method of Twelfth order boundary value problems. The numerical results in the Tables [1.1-1.2], show that the optimal homotopy asymptotic method provides highly accurate numerical results as compared to Homotopy perturbation method. It can be concluded that optimal homotopy asymptotic method is a highly efficient method for solving 12^{th} order boundary value problems arising in various fields of engineering and science.

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