Global Chaos Synchronization of Hyperchaotic Lorenz and Hyperchaotic Chen Systems by Adaptive Control

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***Abstract*—In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical hyperchaotic Lorenz systems (2007), identical hyperchaotic Chen systems (2010) and non-identical hyperchaotic Lorenz and hyperchaotic Chen systems. In this paper, we shall assume that the parameters of both master and slave systems are unknown and we devise adaptive synchronizing schemes using the estimates of parameters for both master and slave systems. Our adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive control method is very effective and convenient to synchronize identical and non-identical hyperchaotic Lorenz and hyperchaotic Chen systems. Numerical simulations are shown to demonstrate the effectiveness of the proposed adaptive synchronization schemes for the hyperchaotic systems addressed in this paper.**

Keywords-chaos; synchronization; adaptive control; hyperchaotic Lorenz system; hyperchaotic Chen system.

I. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1]. Since the seminal work of Pecora and Carroll [2], chaos synchronization has been studied extensively in the last two decades [2-17]. Chaos theory has been applied to a variety of fields like physical systems [3], chemical systems [4], ecological systems [5], secure communications [6-8] etc.

In the recent years, various schemes such as PC method [2], OGY method [9], active control [10-12], adaptive control [13-14], time-delay feedback approach [15], backstepping design method [16], sampled-data feedback synchronization method [17], sliding mode control [18], etc. have been successfully applied for chaos synchronization.

In most of the chaos synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the synchronization is to use the output of the master system to control the slave system so that the output of the slave system tracks the output of the master system asymptotically.

In this paper, we apply adaptive control method to derive new results for the global chaos synchronization of identical hyperchaotic Lorenz systems ([19], 2007), identical hyperchaotic Chen systems ([20], 2010) and non-identical hyperchaotic Lorenz and hyperchaotic Chen systems. We assume that the parameters of the master and slave systems are unknown

This paper has been organized as follows. In Section II, we discuss the adaptive synchronization of identical hyperchaotic Lorenz systems. In Section III, we discuss the adaptive synchronization of identical hyperchaotic Chen systems. In Section IV, we discuss the adaptive synchronization of hyperchaotic Lorenz and hyperchaotic Chen systems. In Section V, we summarize the main results obtained in this paper.

II. Adaptive Synchronization of Identical Hyperchaotic Lorenz Systems

A. Theoretical Results

In this section, we discuss the adaptive synchronization of identical hyperchaotic Lorenz systems ([19], 2006), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lorenz dynamics described by

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_4 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ \dot{x}_4 &= r x_2 x_3 \end{aligned} \tag{1}$$

where x_1, x_2, x_3, x_4 are the states and σ, β, ρ, r are unknown parameters of the system.

As the slave system, we consider the controlled hyperchaotic Lorenz dynamics described by

$$\begin{aligned} \dot{y}_1 &= \sigma(y_2 - y_1) + u_1 \\ \dot{y}_2 &= \rho y_1 - y_2 - y_4 - y_1 y_3 + u_2 \\ \dot{y}_3 &= y_1 y_2 - \beta y_3 + u_3 \\ \dot{y}_4 &= r y_2 y_3 + u_4 \end{aligned} \tag{2}$$

where y_1, y_2, y_3, y_4 are the states and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The four-dimensional system (1) is hyperchaotic when the parameter values are taken as

$$\sigma = 10, \quad \beta = 8/3, \quad \rho = 28 \quad \text{and} \quad r = 0.1$$

The state orbits of the hyperchaotic Lorenz system (1) are shown in Fig. 1.

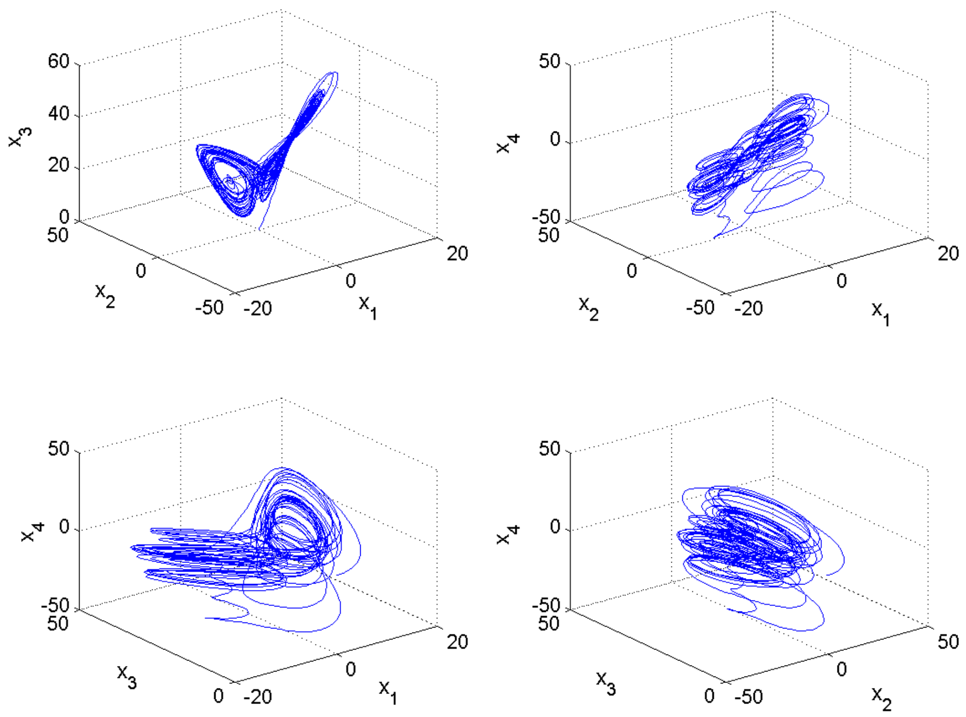

Fig. 1 State Orbits of the Hyperchaotic Lorenz System

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1,2,3,4) \tag{3}$$

The error dynamics is easy obtained as

$$\begin{aligned} \dot{e}_1 &= \sigma(e_2 - e_1) + u_1 \\ \dot{e}_2 &= \rho e_1 - e_2 - e_4 - y_1 y_3 + x_1 x_3 + u_2 \\ \dot{e}_3 &= -\beta e_3 + y_1 y_2 - x_1 x_2 + u_3 \\ \dot{e}_4 &= r(y_2 y_3 - x_2 x_3) + u_4 \end{aligned} \tag{4}$$

Let us now define the adaptive control functions $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ as

$$\begin{aligned} u_1(t) &= -\hat{\sigma}(e_2 - e_1) - k_1 e_1 \\ u_2(t) &= -\hat{\rho} e_1 + e_2 + e_4 + y_1 y_3 - x_1 x_3 - k_2 e_2 \\ u_3(t) &= \hat{\beta} e_3 - y_1 y_2 + x_1 x_2 - k_3 e_3 \\ u_4(t) &= -\hat{r}(y_2 y_3 - x_2 x_3) - k_4 e_4 \end{aligned} \tag{5}$$

where $\hat{\sigma}, \hat{\beta}, \hat{\rho}$ and $\hat{r}$ are estimates of σ, β, ρ and r respectively, and $k_i, (i = 1,2,3,4)$ are positive constants.

Substituting (5) into (4), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (\sigma - \hat{\sigma})(e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= (\rho - \hat{\rho}) e_1 - k_2 e_2 \\ \dot{e}_3 &= -(\beta - \hat{\beta}) e_3 - k_3 e_3 \\ \dot{e}_4 &= (r - \hat{r})(y_2 y_3 - x_2 x_3) - k_4 e_4 \end{aligned} \tag{6}$$

Let us now define the parameter estimation errors as

$$e_\sigma = \sigma - \hat{\sigma},\ e_\beta = \beta - \hat{\beta},\ e_\rho = \rho - \hat{\rho} \ \text{ and } \ e_r = r - \hat{r} \tag{7}$$

Substituting (7) into (6), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_\sigma (e_2 - e_1) - k_1 e_1 \\ \dot{e}_2 &= e_\rho e_1 - k_2 e_2 \\ \dot{e}_3 &= -e_\beta e_3 - k_3 e_3 \\ \dot{e}_4 &= e_r (y_2 y_3 - x_2 x_3) - k_4 e_4 \end{aligned} \tag{8}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V\left(e_1, e_2, e_3, e_4, e_\sigma, e_\beta, e_\rho, e_r\right) = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\sigma^2 + e_\beta^2 + e_\rho^2 + e_r^2\right) \tag{9}$$

which is a positive definite function on R^8.

We also note that

$$\dot{e}_\sigma = -\dot{\hat{\sigma}},\ \ \dot{e}_\beta = -\dot{\hat{\beta}},\ \ \dot{e}_\rho = -\dot{\hat{\rho}} \ \text{ and } \ \dot{e}_r = -\dot{\hat{r}} \tag{10}$$

Differentiating (9) along the trajectories of (8) and using (10), we obtain

$$\begin{aligned} \dot{V} = &-k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 + e_\sigma \left[e_1(e_2 - e_1) - \dot{\hat{\sigma}} \right] + e_\beta \left[-e_3^2 - \dot{\hat{\beta}} \right] \\ &+ e_\rho \left[e_1 e_2 - \dot{\hat{\rho}} \right] + e_r \left[e_4 (y_2 y_3 - x_2 x_3) - \dot{\hat{r}} \right] \end{aligned} \tag{11}$$

In view of Eq. (11), the estimated parameters are updated by the following law:

$$\begin{aligned} \dot{\hat{\sigma}} &= e_1(e_2 - e_1) + k_5 e_\sigma \\ \dot{\hat{\beta}} &= -e_3^2 + k_6 e_\beta \\ \dot{\hat{\rho}} &= e_1 e_2 + k_7 e_\rho \\ \dot{\hat{r}} &= e_4 (y_2 y_3 - x_2 x_3) + k_8 e_r \end{aligned} \tag{12}$$

where k_5, k_6, k_7 and k_8 are positive constants.

Substituting (12) into (11), we obtain

$$\dot{V} = -k_1 e_1^2 - k_2 e_2^2 - k_3 e_3^2 - k_4 e_4^2 - k_5 e_\sigma^2 - k_6 e_\beta^2 - k_7 e_\rho^2 - k_8 e_r^2 \tag{13}$$

which is a negative definite function on R^8.

Thus, by Lyapunov stability theory [21], it is immediate that the synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation error $e_\sigma, e_\beta, e_\rho, e_r$ decay to zero exponentially with time. Thus, it follows that the master system (1) and the slave system (2) are completely synchronized and that the parameter estimates converges to the original values of the system parameters.

Hence, we have proved the following result.

Theorem 1. The identical hyperchaotic Lorenz systems (1) and (2) with unknown parameters are globally and exponentially synchronized by the adaptive control law (5), where the update law for the parameter estimates is given by (12) and $k_i, (i = 1, \ldots, 8)$ are positive constants.■

B. Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (1) and (2) with the adaptive control law (5) and the parameter update law (12) using MATLAB.

For the hyperchaotic Lorenz systems (1) and (2), the parameter values are taken as

$$\sigma = 10, \quad \beta = 8/3, \quad \rho = 28 \quad \text{and} \quad r = 0.1$$

Suppose that the initial values of the parameter estimates are

$$\hat{\sigma}(0) = 2, \quad \hat{\beta}(0) = 20, \quad \hat{\rho}(0) = 10 \quad \text{and} \quad \hat{r}(0) = 5.$$

The initial values of the master system (1) are taken as

$$x_1(0) = 18, \quad x_2(0) = 24, \quad x_3(0) = 26 \quad \text{and} \quad x_4(0) = 40.$$

The initial values of the slave system (2) are taken as

$$y_1(0) = 29, \quad y_2(0) = 6, \quad y_3(0) = 45 \text{ and } y_4(0) = 15.$$

Fig. 2 depicts the complete synchronization of the identical hyperchaotic systems (1) and (2).

Fig. 3 shows that the estimated values of the parameters, viz. $\hat{\sigma}, \hat{\beta}, \hat{\rho}$ and $\hat{r}$ converge to the system parameters $\sigma = 10, \ \beta = 8/3, \ \rho = 28$ and $r = 0.1$, respectively.

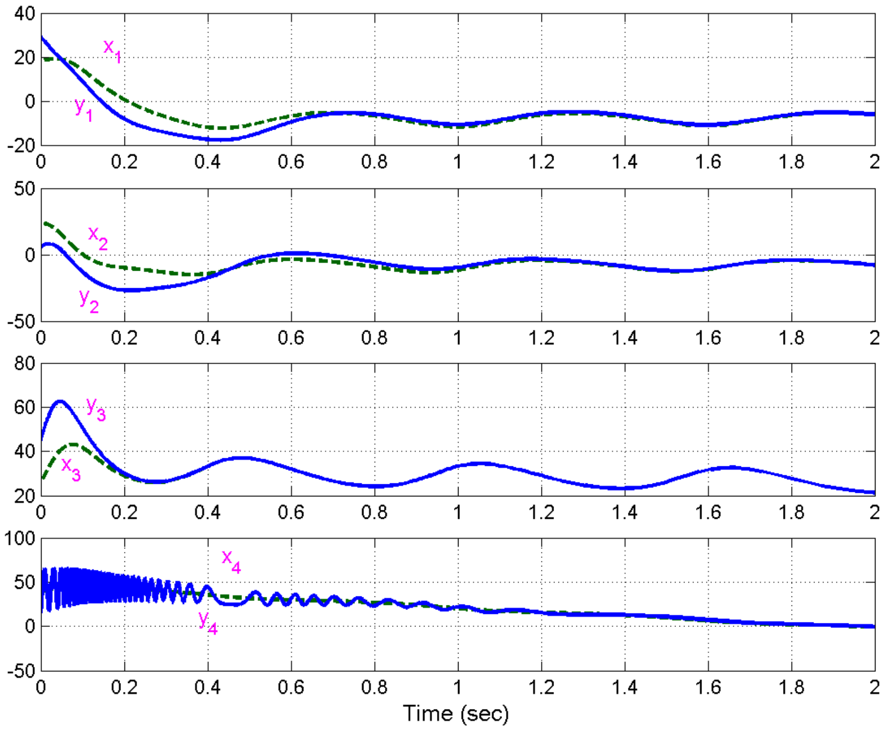

Fig. 2 Synchronization of the Identical Hyperchaotic Lorenz Systems

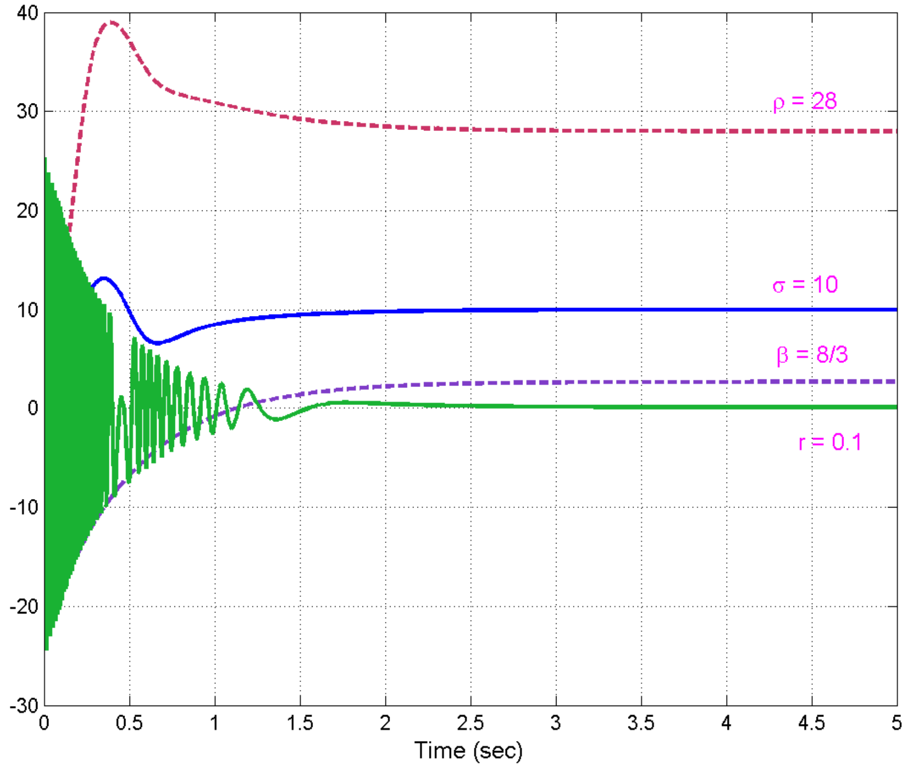

Fig. 3 Parameter Esimates $\hat{\sigma}(t), \hat{\beta}(t), \hat{\rho}(t), \hat{r}(t)$

III. Adaptive Synchronization of Identical Hyperchaotic Chen Systems

A. *Theoretical Results*

In this section, we discuss the adaptive synchronization of identical hyperchaotic Chen systems ([20], 2008), where the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Chen dynamics described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= 4x_1 - 10x_1x_3 + cx_2 + 4x_4 \\ \dot{x}_3 &= x_2^2 - bx_3 \\ \dot{x}_4 &= -dx_1 \end{aligned} \tag{14}$$

where x_1, x_2, x_3, x_4 are the states and a, b, c, d are unknown parameters of the system.

As the slave system, we consider the controlled hyperchaotic Chen dynamics described by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + u_1 \\ \dot{y}_2 &= 4y_1 - 10y_1y_3 + cy_2 + 4y_4 + u_2 \\ \dot{y}_3 &= y_2^2 - by_3 + u_3 \\ \dot{y}_4 &= -dy_1 + u_4 \end{aligned} \tag{15}$$

where y_1, y_2, y_3, y_4 are the states and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The four-dimensional system (14) is hyperchaotic when the parameter values are taken as

$$a = 35, \quad b = 3, \quad c = 21 \quad \text{and} \quad d = 2.$$

The state orbits of the hyperchaotic Chen system (14) are shown in Fig. 4.

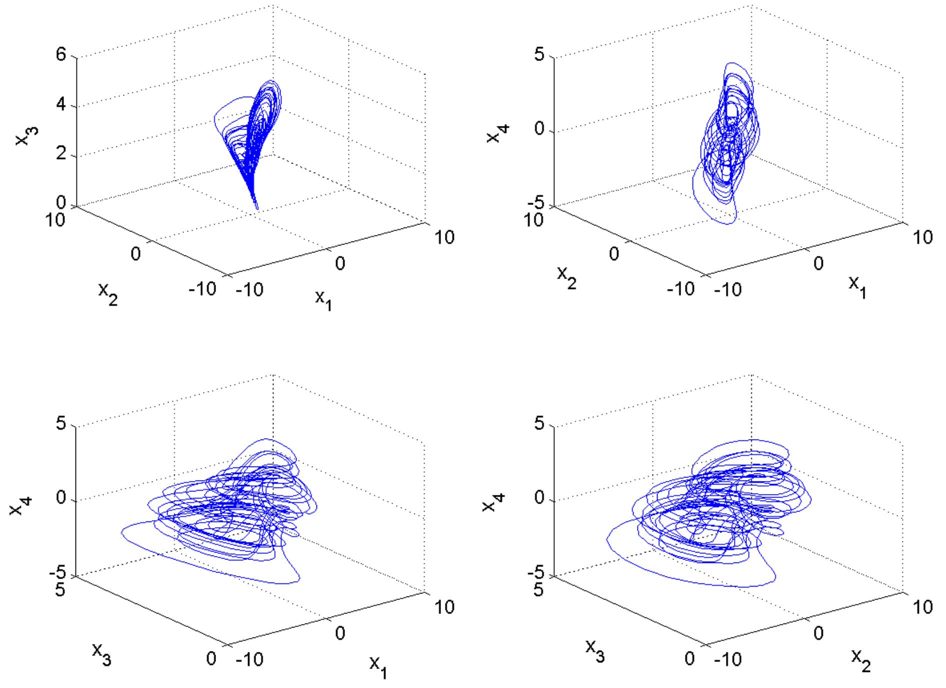

Fig. 4 State Orbits of the Hyperchaotic Chen System

The chaos synchronization error is defined by

$$e_i = y_i - x_i, \quad (i = 1, 2, 3, 4) \tag{16}$$

The error dynamics is easy obtained as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + u_1 \\ \dot{e}_2 &= 4e_1 - 10(y_1 y_3 - x_1 x_3) + c e_2 + 4e_4 + u_2 \\ \dot{e}_3 &= -b e_3 + y_2^2 - x_2^2 + u_3 \\ \dot{e}_4 &= -d e_1 + u_4 \end{aligned} \tag{17}$$

Let us now define the adaptive control functions $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ as

$$\begin{aligned} u_1(t) &= -\hat{a}(e_2 - e_1) - k_1 e_1 \\ u_2(t) &= -4e_1 + 10(y_1 y_3 - x_1 x_3) - \hat{c} e_2 - 4e_4 - k_2 e_2 \\ u_3(t) &= \hat{b} e_3 - y_2^2 + x_2^2 - k_3 e_3 \\ u_4(t) &= \hat{d} e_1 - k_4 e_4 \end{aligned} \tag{18}$$

where $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are estimates of a, b, c and d respectively, and $k_i, (i = 1, 2, 3, 4)$ are positive constants.

Substituting (18) into (17), the error dynamics simplifies to

$$\begin{aligned}\dot{e}_1 &= (a-\hat{a})(e_2-e_1)-k_1e_1\\ \dot{e}_2 &= (c-\hat{c})e_2-k_2e_2\\ \dot{e}_3 &= -(b-\hat{b})e_3-k_3e_3\\ \dot{e}_4 &= -(d-\hat{d})e_1-k_4e_4\end{aligned} \tag{19}$$

Let us now define the parameter estimation errors as

$$e_a = a-\hat{a},\ e_b = b-\hat{b},\ e_c = c-\hat{c} \text{ and } e_d = d-\hat{d} \tag{20}$$

Substituting (20) into (19), we obtain the error dynamics as

$$\begin{aligned}\dot{e}_1 &= e_a(e_2-e_1)-k_1e_1\\ \dot{e}_2 &= e_ce_2-k_2e_2\\ \dot{e}_3 &= -e_be_3-k_3e_3\\ \dot{e}_4 &= -e_de_1-k_4e_4\end{aligned} \tag{21}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V\left(e_1,e_2,e_3,e_4,e_a,e_b,e_c,e_d\right) = \frac{1}{2}\left(e_1^2+e_2^2+e_3^2+e_4^2+e_a^2+e_b^2+e_c^2+e_d^2\right) \tag{22}$$

which is a positive definite function on R^8.

We also note that

$$\dot{e}_a = -\dot{\hat{a}},\ \ \dot{e}_b = -\dot{\hat{b}},\ \ \dot{e}_c = -\dot{\hat{c}} \text{ and } \dot{e}_d = -\dot{\hat{d}} \tag{23}$$

Differentiating (22) along the trajectories of (21) and using (23), we obtain

$$\begin{aligned}\dot{V} = &-k_1e_1^2-k_2e_2^2-k_3e_3^2-k_4e_4^2+e_a\left[e_1(e_2-e_1)-\dot{\hat{a}}\right]+e_b\left[-e_3^2-\dot{\hat{b}}\right]\\ &+e_c\left[e_2^2-\dot{\hat{c}}\right]+e_d\left[-e_1e_4-\dot{\hat{d}}\right]\end{aligned} \tag{24}$$

In view of Eq. (24), the estimated parameters are updated by the following law:

$$\begin{aligned}\dot{\hat{a}} &= e_1(e_2-e_1)+k_5e_a\\ \dot{\hat{b}} &= -e_3^2+k_6e_b\\ \dot{\hat{c}} &= e_2^2+k_7e_c\\ \dot{\hat{d}} &= -e_1e_4+k_8e_d\end{aligned} \tag{25}$$

where k_5, k_6, k_7 and k_8 are positive constants.

Substituting (25) into (24), we obtain

$$\dot{V} = -k_1e_1^2-k_2e_2^2-k_3e_3^2-k_4e_4^2-k_5e_a^2-k_6e_b^2-k_7e_c^2-k_8e_d^2 \tag{26}$$

which is a negative definite function on R^8.

Thus, by Lyapunov stability theory [21], it is immediate that the synchronization error $e_i, (i = 1,2,3,4)$ and the parameter estimation error e_a, e_b, e_c, e_d decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 2. The identical hyperchaotic Chen systems (14) and (15) with unknown parameters are globally and exponentially synchronized by the adaptive control law (18), where the update law for the parameter estimates is given by (25) and $k_i, (i = 1, \ldots, 8)$ are positive constants.■

B. Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (14) and (15) with the adaptive control law (18) and the parameter update law (25) using MATLAB.

For the hyperchaotic Chen systems (14) and (15), the parameter values are taken as

$$a = 35, \quad b = 3, \quad c = 21 \quad \text{and} \quad d = 2.$$

Suppose that the initial values of the parameter estimates are

$$\hat{a}(0) = 12, \quad b(0) = 8, \quad c(0) = 20 \quad \text{and} \quad \hat{d}(0) = 4.$$

The initial values of the master system (14) are taken as

$$x_1(0) = 26, \quad x_2(0) = 35, \quad x_3(0) = 42 \quad \text{and} \quad x_4(0) = 10.$$

The initial values of the slave system (15) are taken as

$$y_1(0) = 19, \quad y_2(0) = 26, \quad y_3(0) = 15 \quad \text{and} \quad y_4(0) = 22.$$

Fig. 5 depicts the complete synchronization of the identical hyperchaotic systems (14) and (15).

Fig. 6 shows that the estimated values of the parameters, viz. $\hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ converge to the system parameters $a = 35, \; b = 3, \; c = 21$ and $d = 2,$ respectively.

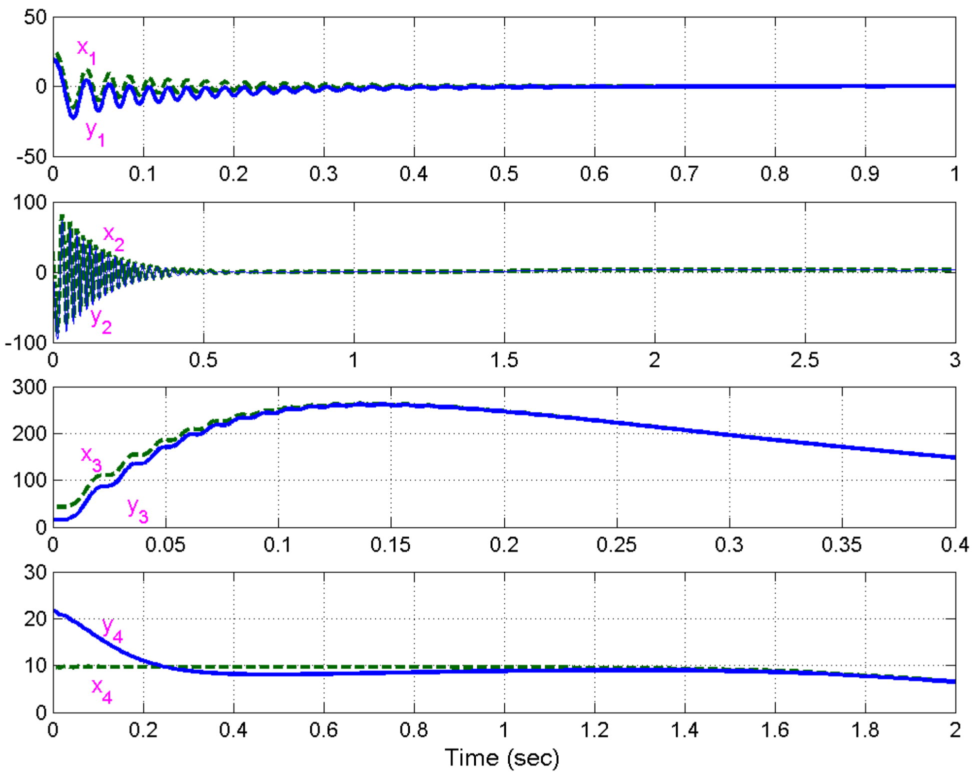

Fig. 5 Synchronization of the Identical Hyperchaotic Chen Systems

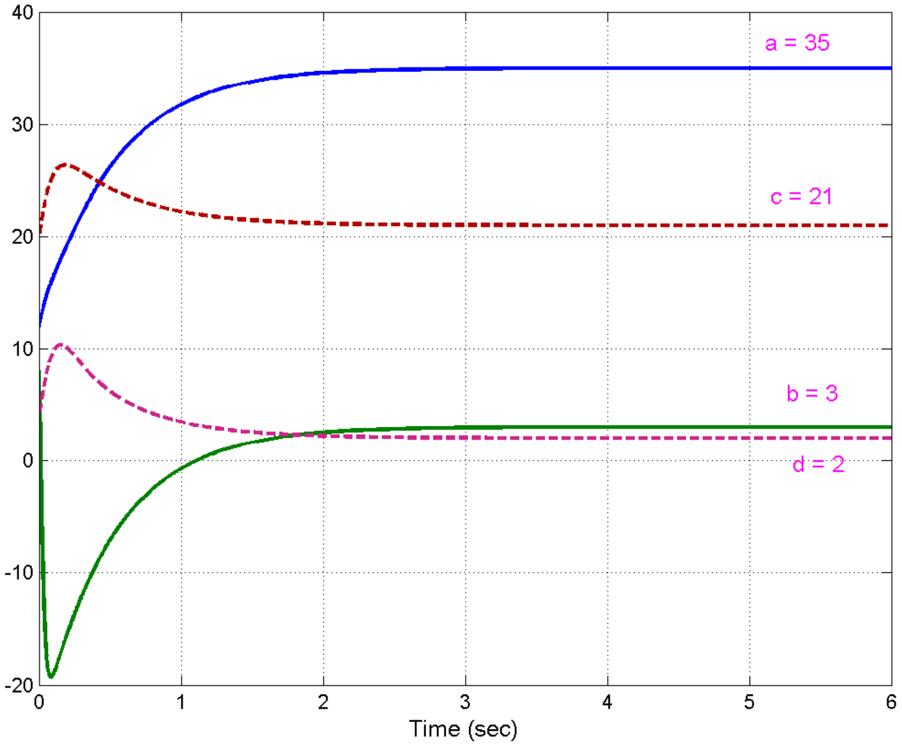

Fig. 6 Parameter Estimates $\hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$

IV. Adaptive Synchronization of Hyperchaotic Lorenz and hyperchaotic Chen Systems

A. *Theoretical Results*

In this section, we discuss the adaptive synchronization of non-identical hyperchaotic Lorenz system ([19], 2006) and hyperchaotic Chen system ([20], 2010). We consider the hyperchaotic Lorenz system as the master system and the hyperchaotic Chen system as the slave system and assume that the parameters of the master and slave systems are unknown.

As the master system, we consider the hyperchaotic Lorenz dynamics described by

$$\begin{aligned} \dot{x}_1 &= \sigma(x_2 - x_1) \\ \dot{x}_2 &= \rho x_1 - x_2 - x_4 - x_1 x_3 \\ \dot{x}_3 &= x_1 x_2 - \beta x_3 \\ \dot{x}_4 &= r x_2 x_3 \end{aligned} \tag{27}$$

where x_1, x_2, x_3, x_4 are the states and σ, β, ρ, r are unknown parameters of the system.

As the slave system, we consider the controlled hyperchaotic Chen dynamics described by

$$\begin{aligned} \dot{y}_1 &= a(y_2 - y_1) + u_1 \\ \dot{y}_2 &= 4y_1 - 10 y_1 y_3 + c y_2 + 4 y_4 + u_2 \\ \dot{y}_3 &= y_2^2 - b y_3 + u_3 \\ \dot{y}_4 &= -d y_1 + u_4 \end{aligned} \tag{28}$$

where y_1, y_2, y_3, y_4 are the states, a, b, c, d are unknown parameters of the system and u_1, u_2, u_3, u_4 are the nonlinear controllers to be designed.

The chaos synchronization error is defined by

$$\begin{aligned} e_1 &= y_1 - x_1 \\ e_2 &= y_2 - x_2 \\ e_3 &= y_3 - x_3 \\ e_4 &= y_4 - x_4 \end{aligned} \tag{29}$$

The error dynamics is easy obtained as

$$\begin{aligned} \dot{e}_1 &= a(y_2 - y_1) - \sigma(x_2 - x_1) + u_1 \\ \dot{e}_2 &= 4y_1 - 10y_1y_3 + cy_2 + 4y_4 - \rho x_1 + x_2 + x_4 + x_1x_3 + u_2 \\ \dot{e}_3 &= y_2^2 - by_3 - x_1x_2 + \beta x_3 + u_3 \\ \dot{e}_4 &= -dy_1 - rx_2x_3 + u_4 \end{aligned} \tag{30}$$

Let us now define the adaptive control functions $u_1(t), u_2(t), u_3(t)$ and $u_4(t)$ as

$$\begin{aligned} u_1(t) &= -\hat{a}(y_2 - y_1) + \hat{\sigma}(x_2 - x_1) - k_1e_1 \\ u_2(t) &= -4y_1 + 10y_1y_3 - \hat{c}y_2 - 4y_4 + \hat{\rho}x_1 - x_2 - x_4 - x_1x_3 - k_2e_2 \\ u_3(t) &= -y_2^2 + \hat{b}y_3 + x_1x_2 - \hat{\beta}x_3 - k_3e_3 \\ u_4(t) &= \hat{d}y_1 + \hat{r}x_2x_3 - k_4e_4 \end{aligned} \tag{31}$$

where $\hat{\sigma}, \hat{\beta}, \hat{\rho}, \hat{r}, \hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ are estimates of $\sigma, \beta, \rho, r, a, b, c$ and d respectively, and k_i, $(i = 1, 2, 3, 4)$ are positive constants.

Substituting (31) into (30), the error dynamics simplifies to

$$\begin{aligned} \dot{e}_1 &= (a - \hat{a})(y_2 - y_1) - (\sigma - \hat{\sigma})(x_2 - x_1) - k_1e_1 \\ \dot{e}_2 &= (c - \hat{c})y_2 - (\rho - \hat{\rho})x_1 - k_2e_2 \\ \dot{e}_3 &= -(b - \hat{b})y_3 + (\beta - \hat{\beta})x_3 - k_3e_3 \\ \dot{e}_4 &= -(d - \hat{d})y_1 - (r - \hat{r})x_2x_3 - k_4e_4 \end{aligned} \tag{32}$$

Let us now define the parameter estimation errors as

$$\begin{aligned} &e_\sigma = \sigma - \hat{\sigma},\ e_\beta = \beta - \hat{\beta},\ e_\rho = \rho - \hat{\rho},\ e_r = r - \hat{r} \\ &e_a = a - \hat{a},\ e_b = b - \hat{b},\ e_c = c - \hat{c},\ \ e_d = d - \hat{d} \end{aligned} \tag{33}$$

Substituting (33) into (32), we obtain the error dynamics as

$$\begin{aligned} \dot{e}_1 &= e_a(y_2 - y_1) - e_\sigma(x_2 - x_1) - k_1e_1 \\ \dot{e}_2 &= e_cy_2 - e_\rho x_1 - k_2e_2 \\ \dot{e}_3 &= -e_by_3 + e_\beta x_3 - k_3e_3 \\ \dot{e}_4 &= -e_dy_1 - e_rx_2x_3 - k_4e_4 \end{aligned} \tag{34}$$

For the derivation of the update law for adjusting the estimates of the parameters, the Lyapunov approach is used.

We consider the quadratic Lyapunov function defined by

$$V = \frac{1}{2}\left(e_1^2 + e_2^2 + e_3^2 + e_4^2 + e_\sigma^2 + e_\beta^2 + e_\rho^2 + e_r^2 + e_a^2 + e_b^2 + e_c^2 + e_d^2\right), \tag{35}$$

which is a positive definite function on R^{12}.

We also note that

$$\dot{e}_\sigma = -\dot{\hat{\sigma}},\ \ \dot{e}_\beta = -\dot{\hat{\beta}},\ \ \dot{e}_\rho = -\dot{\hat{\rho}},\ \ \dot{e}_r = -\dot{\hat{r}},\ \dot{e}_a = -\dot{\hat{a}},\ \ \dot{e}_b = -\dot{\hat{b}},\ \ \dot{e}_c = -\dot{\hat{c}},\ \ \dot{e}_d = -\dot{\hat{d}} \tag{36}$$

Differentiating (35) along the trajectories of (34) and using (36), we obtain

$$\begin{aligned}\dot{V} &= -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 + e_\sigma\left[-e_1(x_2 - x_1) - \dot{\hat{\sigma}}\right] + e_\beta\left[e_3x_3 - \dot{\hat{\beta}}\right] \\ &\quad + e_\rho\left[-e_2x_1 - \dot{\hat{\rho}}\right] + e_r\left[-e_4x_2x_3 - \dot{\hat{r}}\right] + e_a\left[e_1(y_2 - y_1) - \dot{\hat{a}}\right] \\ &\quad + e_b\left[-e_3y_3 - \dot{\hat{b}}\right] + e_c\left[e_2y_2 - \dot{\hat{c}}\right] + e_d\left[-e_4y_1 - \dot{\hat{d}}\right]\end{aligned} \tag{37}$$

In view of Eq. (24), the estimated parameters are updated by the following law:

$$\begin{aligned} &\dot{\hat{\sigma}} = -e_1(x_2 - x_1) + k_5e_\sigma, && \dot{\hat{a}} = e_1(y_2 - y_1) + k_9e_a \\ &\dot{\hat{\beta}} = e_3x_3 + k_6e_\beta, && \dot{\hat{b}} = -e_3y_3 + k_{10}e_b \\ &\dot{\hat{\rho}} = -e_2x_1 + k_7e_\rho, && \dot{\hat{c}} = e_2y_2 + k_{11}e_c \\ &\dot{\hat{r}} = -e_4x_2x_3 + k_8e_r, && \dot{\hat{d}} = -e_4y_1 + k_{12}e_d \end{aligned} \tag{38}$$

where $k_i, (i = 5, \ldots, 12)$ are positive constants.

Substituting (38) into (37), we obtain

$$\dot{V} = -k_1e_1^2 - k_2e_2^2 - k_3e_3^2 - k_4e_4^2 - k_5e_\sigma^2 - k_6e_\beta^2 - k_7e_\rho^2 - k_8e_r^2 - k_9e_a^2 - k_{10}e_b^2 - k_{11}e_c^2 - k_{12}e_d^2 \tag{39}$$

which is a negative definite function on R^{12}.

Thus, by Lyapunov stability theory [21], it is immediate that the synchronization error $e_i, (i = 1, 2, 3, 4)$ and the parameter estimation errors $e_\sigma, e_\beta, e_\rho, e_r, e_a, e_b, e_c, e_d$ decay to zero exponentially with time.

Hence, we have proved the following result.

Theorem 3. The non-identical hyperchaotic Lorenz system (27) and hyperchaotic Chen system (28) with unknown parameters are globally and exponentially synchronized by the adaptive control law (31), where the update law for the parameter estimates is given by (38) and $k_i, (i = 1, \ldots, 12)$ are positive constants.■

B. Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic systems (27) and (28) with the adaptive control law (31) and the parameter update law (38) using MATLAB. For the hyperchaotic Lorenz system (27) and hyperchaotic Chen system (28), the parameter values are taken as

$$\sigma = 10,\ \ \beta = 8/3,\ \ \rho = 28,\ r = 0.1,\ a = 35,\ \ b = 3,\ \ c = 21\ \text{ and }\ d = 2.$$

Suppose that the initial values of the parameter estimates are

$$\hat{\sigma}(0) = 5,\ \hat{\beta}(0) = 2,\ \hat{\rho}(0) = 4,\ \hat{r}(0) = 6,\ \hat{a}(0) = 10,\ \hat{b}(0) = 5,\ \hat{c}(0) = 4\ \text{ and }\ \hat{d}(0) = 8.$$

The initial values of the master system (27) are taken as

$$x_1(0) = 19,\ \ x_2(0) = 42,\ \ x_3(0) = 38\ \text{ and }\ x_4(0) = 30.$$

The initial values of the slave system (28) are taken as

$y_1(0) = 28,\ \ y_2(0) = 12,\ \ y_3(0) = 46$ and $y_4(0) = 17.$

Fig. 7 depicts the complete synchronization of the non-identical hyperchaotic systems (27) and (28).

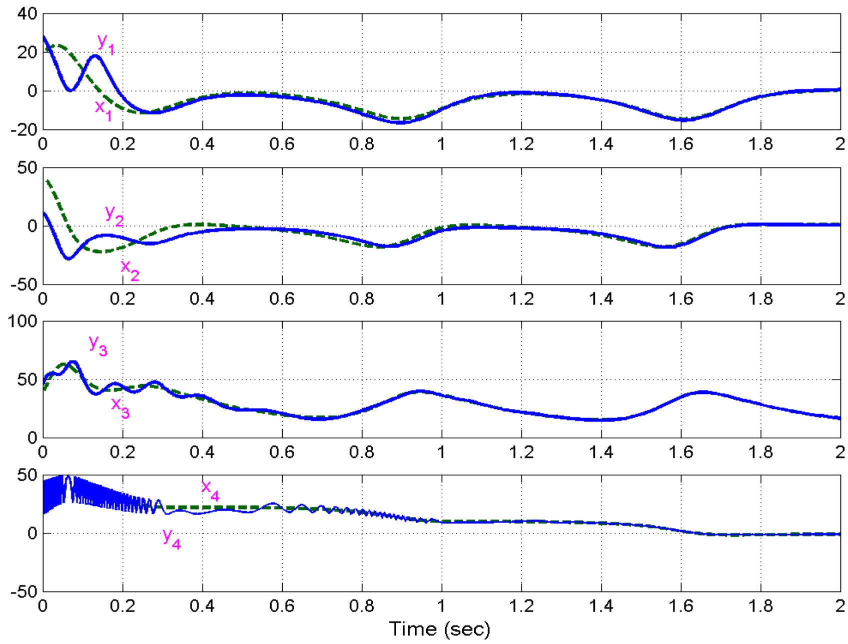

Fig. 7 Synchronization of the Hyperchaotic Lorenz and Hyperchaotic Chen Systems

Fig. 8 shows that the estimated values of the parameters, viz. $\hat{\sigma},\ \hat{\beta},\ \hat{\rho}, \hat{r}, \hat{a}, \hat{b}, \hat{c}$ and $\hat{d}$ converge to the system parameters $\sigma = 10,\ \ \beta = 8/3,\ \ \rho = 28,\ r = 0.1,\ a = 35,\ b = 3,\ c = 21$ and $d = 2,$ respectively.

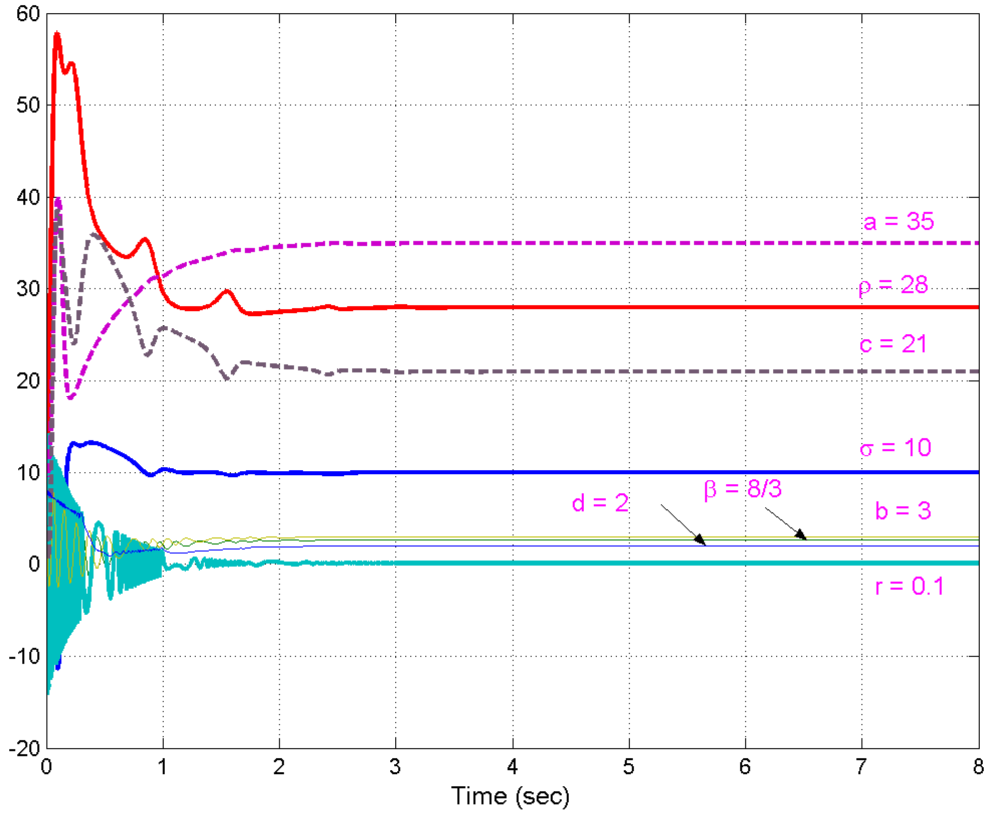

Fig. 8 Parameter Estimates $\hat{\sigma}(t), \hat{\beta}(t), \hat{\rho}(t), \hat{r}(t), \hat{a}(t), \hat{b}(t), \hat{c}(t), \hat{d}(t)$

V. CONCLUSIONS

In this paper, we have applied adaptive control method for the global chaos synchronization of identical hyperchaotic Lorenz systems (2006), identical hyperchaotic Chen systems (2010) and non-identical hyperchaotic Lorenz and Chen systems with unknown parameters. The adaptive synchronization results derived in this paper are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the adaptive nonlinear control method is very effective and convenient to achieve global chaos synchronization for the uncertain hyperchaotic systems discussed in this paper. Numerical simulations are also shown for the synchronization of identical and non-identical uncertain hyperchaotic Lorenz and hyperchaotic Chen systems to demonstrate the effectiveness of the adaptive synchronization schemes derived in this paper.

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