Anti-Synchronization of the Hyperchaotic Lorenz Systems by Sliding Mode Control

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***Abstract*—This paper investigates the problem of global chaos anti-synchronization of identical hyperchaotic Lorenz systems (Jia, 2007) by sliding mode control. The stability results derived in this paper for the anti-synchronization of identical hyperchaotic Lorenz systems are established using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the sliding mode control method is very effective and convenient to achieve global chaos synchronization of the identical hyperchaotic Lorenz systems. Numerical simulations are shown to illustrate the effectiveness of the synchronization schemes derived in this paper.**

Keywords-nonlinear control systems; chaos; anti-synchronization; sliding mode control; hyperchaotic Lorenz system.

I. INTRODUCTION

Chaotic systems are dynamical systems that are highly sensitive to initial conditions. The sensitive nature of chaotic systems is commonly called as the *butterfly effect* [1].

Synchronization of chaotic systems is a phenomenon which may occur when two or more chaotic oscillators are coupled or when a chaotic oscillator drives another chaotic oscillator. Because of the butterfly effect which causes the exponential divergence of the trajectories of two identical chaotic systems started with nearly the same initial conditions, synchronizing two chaotic systems is seemingly a very challenging problem.

In most of the chaos anti-synchronization approaches, the *master-slave* or *drive-response* formalism is used. If a particular chaotic system is called the *master* or *drive* system and another chaotic system is called the *slave* or *response* system, then the idea of the anti-synchronization is to use the output of the master system to control the slave system so that the outputs of the slave system have the same amplitude but opposite signs as the outputs of the master system asymptotically. In other words, the sum of the outputs of the master and slave systems are expected to converge to zero asymptotically when anti-synchronization appears.

Since the pioneering work by Pecora and Carroll ([2], 1990), chaos synchronization problem has been studied extensively and intensively in the literature [2-17]. Chaos theory has been applied to a variety of fields such as physical systems [3], chemical systems [4], ecological systems [5], secure communications [6-8], etc.

In the last two decades, various schemes have been successfully applied for chaos synchronization such as PC method [2], OGY method [9], active control method [10-12], adaptive control method [13-15], time-delay feedback method [16], backstepping design method [17], sampled-data feedback method [18], etc. Recently, active control has been applied to anti-synchronize identical chaotic systems [19-20] and different hyperchaotic systems [21].

In this paper, we derive new control results based on the sliding mode control [22-24] for the global chaos anti-synchronization of identical hyperchaotic Lorenz systems (Jia, [25], 2007).

In robust control systems, the sliding control method is often adopted due to its inherent advantages of easy realization, fast response and good transient performance as well as its insensitivity to parameter uncertainties and external disturbances.

This paper has been organized as follows. In Section II, we describe the problem statement and our methodology using sliding mode control. In Section III, we discuss the global chaos anti-synchronization of identical hyperchaotic Lorenz systems. In Section IV, we summarize the main results obtained in this paper.

II. Problem Statement and Our Methodology using Sliding Mode Control

In this section, we describe the problem statement for the global chaos synchronization for identical chaotic systems and our methodology using sliding control.

Consider the chaotic system described by

$$\dot{x} = Ax + f(x) \tag{1}$$

where $x \in R^n$ is the state of the system, A is the $n \times n$ matrix of the system parameters and $f : R^n \to R^n$ is the nonlinear part of the system. We consider the system (1) as the *master* or *drive* system.

As the *slave* or *response* system, we consider the following chaotic system described by the dynamics

$$\dot{y} = Ay + f(y) + u \tag{2}$$

where $y \in R^n$ is the state of the system and $u \in R^m$ is the controller to be designed.

If we define the *anti-synchronization error* as

$$e = y + x, \tag{3}$$

then the error dynamics is obtained as

$$\dot{e} = Ae + \eta(x, y) + u, \tag{4}$$

where

$$\eta(x, y) = f(y) + f(x) \tag{5}$$

The objective of the global chaos anti-synchronization problem is to find a controller u such that

$$\lim_{t \to \infty} \| e(t) \| = 0 \quad \text{for all } e(0) \in R^n.$$

To solve this problem, we first define the control u as

$$u = -\eta(x, y) + Bv \tag{6}$$

where B is a constant gain vector selected such that (A, B) is controllable.

Substituting (5) into (4), the error dynamics simplifies to

$$\dot{e} = Ae + Bv \tag{7}$$

which is a linear time-invariant control system with single input v.

Thus, the original global chaos synchronization problem can be replaced by an equivalent problem of stabilizing the zero solution $e = 0$ of the system (7) by a suitable choice of the sliding control.

In the sliding control, we define the variable

$$s(e) = Ce = c_1 e_1 + c_2 e_2 + \cdots + c_n e_n \tag{8}$$

In the sliding control, we constrain the motion of the system (7) to the sliding manifold defined by

$$S = \left\{ x \in R^n \mid s(e) = 0 \right\}$$

which is required to be invariant under the flow of the error dynamics (7).

When in sliding manifold S, the system (7) satisfies the following conditions:

$$s(e) = 0 \tag{9}$$

which is the defining equation for the manifold S and

$$\dot{s}(e) = 0 \tag{10}$$

which is the necessary condition for the state trajectory $e(t)$ of (7) to stay on the sliding manifold S.

Using (7) and (8), the equation (10) can be rewritten as

$$\dot{s}(e) = C\left[Ae + Bv\right] = 0 \tag{11}$$

Solving (11) for v, we obtain the equivalent control law

$$v_{\text{eq}}(t) = -(CB)^{-1}CA\, e(t) \tag{12}$$

where C is chosen such that $CB \neq 0$.

Substituting (12) into the error dynamics (7), we obtain the closed-loop dynamics as

$$\dot{e} = \left[I - B(CB)^{-1}C\right]Ae \tag{13}$$

The row vector C is selected such that the system matrix of the controlled dynamics $\left[I - B(CB)^{-1}C\right]A$ is Hurwitz. Then the system (13) is globally asymptotically stable.

To design the sliding controller for (7), we apply the constant plus proportional rate reaching law

$$\dot{s} = -q\,\text{sgn}(s) - k\, s \tag{14}$$

where $\text{sgn}(\cdot)$ denotes the sign function and the gains $q > 0,\ k > 0$ are determined such that the sliding condition is satisfied and sliding motion will occur.

From equations (11) and (14), we can obtain the control $v(t)$ as

$$v(t) = -(CB)^{-1}\left[C(kI + A)e + q\,\text{sgn}(s)\right] \tag{15}$$

which yields

$$v(t) = \begin{cases} -(CB)^{-1}\left[C(kI + A)e + q\right], & \text{if } s(e) > 0 \\ -(CB)^{-1}\left[C(kI + A)e - q\right], & \text{if } s(e) < 0 \end{cases} \tag{16}$$

Theorem 1. The master system (1) and the slave system (2) are globally and asymptotically anti-synchronized for all initial conditions $x(0), y(0) \in R^n$ by the feedback control law

$$u(t) = -\eta(x, y) + Bv(t) \tag{17}$$

where $v(t)$ is defined by (15) and B is a column vector such that (A, B) is controllable. Also, the sliding mode gains k, q are positive.

Proof. First, we note that substituting (17) and (15) into the error dynamics (4), we obtain the closed-loop error dynamics as

$$\dot{e} = Ae - B(CB)^{-1}\left[C(kI + A)e + q\,\text{sgn}(s)\right] \tag{18}$$

To prove that the error dynamics (18) is globally asymptotically stable, we consider the Lyapunov function defined by the equation

$$V(e) = \frac{1}{2}s^2(e) \tag{19}$$

Clearly, V is a positive definite function on R^n.

Differentiating V along the trajectories of (18) or the equivalent dynamics (14), we get

$$\dot{V}(e) = s(e)\dot{s}(e) = -ks^2 - q\,\mathrm{sgn}(s)s \tag{20}$$

which is a negative definite function on R^n.

This calculation shows that V is a globally defined, positive definite, Lyapunov function for the error dynamics (18), which has a globally defined, negative definite time derivative $\dot{V}$.

Thus, by Lyapunov stability theory [22], it is immediate that the error dynamics (18) is globally asymptotically stable for all initial conditions $e(0) \in R^n$.

This means that for all initial conditions $e(0) \in R^n$, we have

$$\lim_{t\to\infty} \|e(t)\| = 0$$

Hence, it follows that the master system (1) and the slave system (2) are globally and asymptotically synchronized for all initial conditions $x(0), y(0) \in R^n$.

This completes the proof. ■

III. Anti- Synchronization of Identical Hyperchaotic Lorenz Systems

A. *Theoretical Results*

In this section, we apply the sliding mode control results derived in Section 2 for the anti-synchronization of identical hyperchaotic Lorenz systems ([25], 2007).

The hyperchaotic Lorenz system is one of the paradigms of the four-dimensional hyperchaotic systems discovered by G. Jia ([25], 2007).

Thus, the master system is described by the hyperchaotic Lorenz dynamics

$$\begin{aligned}
\dot{x}_1 &= a(x_2 - x_1) + x_4 \\
\dot{x}_2 &= -x_1x_3 + rx_1 - x_2 \\
\dot{x}_3 &= x_1x_2 - bx_3 \\
\dot{x}_4 &= -x_1x_3 + dx_4
\end{aligned} \tag{21}$$

where x_1, x_2, x_3, x_4 are state variables of the system and a, b, r, d are positive, constant parameters of the system.

The slave system is described by the controlled hyperchaotic Lorenz dynamics

$$\begin{aligned}
\dot{y}_1 &= a(y_2 - y_1) + y_4 + u_1 \\
\dot{y}_2 &= -y_1y_3 + ry_1 - y_2 + u_2 \\
\dot{y}_3 &= y_1y_2 - by_3 + u_3 \\
\dot{y}_4 &= -y_1y_3 + dy_4 + u_4
\end{aligned} \tag{22}$$

where y_1, y_2, y_3, y_4 are state variables and u_1, u_2, u_3, u_4 are the controllers to be designed.

The four-dimensional system (21) is hyperchaotic when the parameter values are taken as

$$a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.$$

The hyperchaotic portrait of the Lorenz system (21) is illustrated in Fig. 1.

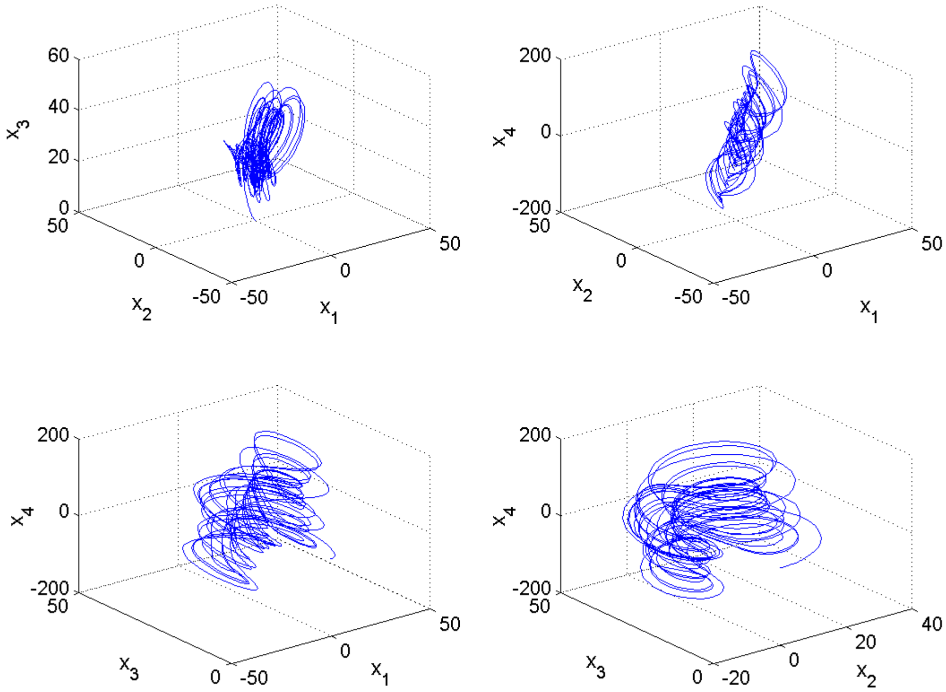

Figure 1. State Portrait of the Hyperchaotic Lorenz System

The anti-synchronization error is defined by

$$e_i = y_i + x_i, \ (i = 1, 2, 3, 4) \tag{23}$$

The error dynamics is easily obtained as

$$\begin{aligned} \dot{e}_1 &= a(e_2 - e_1) + e_4 + u_1 \\ \dot{e}_2 &= re_1 - e_2 - y_1 y_3 - x_1 x_3 + u_2 \\ \dot{e}_3 &= -be_3 + y_1 y_2 + x_1 x_2 + u_3 \\ \dot{e}_4 &= de_4 - y_1 y_3 - x_1 x_3 + u_4 \end{aligned} \tag{24}$$

We write the error dynamics (24) in the matrix notation as

$$\dot{e} = Ae + \eta(x, y) + u \tag{25}$$

where

$$A = \begin{bmatrix} -a & a & 0 & 1 \\ r & -1 & 0 & 0 \\ 0 & 0 & -b & 0 \\ 0 & 0 & 0 & d \end{bmatrix}, \quad \eta(x, y) = \begin{bmatrix} 0 \\ -y_1 y_3 - x_1 x_3 \\ y_1 y_2 + x_1 x_2 \\ -y_1 y_3 - x_1 x_3 \end{bmatrix} \text{ and } u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}. \tag{26}$$

The sliding mode controller design is carried out as detailed in Section 2.

First, we set u as

$$u = -\eta(x, y) + Bv \tag{27}$$

where B is chosen such that (A, B) is controllable.

We take B as

$$B = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}. \tag{28}$$

In the hyperchaotic case, the parameter values are

$$a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.$$

The sliding mode variable is selected as

$$s = Ce = \begin{bmatrix} -3 & -3 & 0 & 2 \end{bmatrix} e = -3e_1 - 3e_2 + 2e_4 \tag{29}$$

which makes the sliding mode state equation asymptotically stable.

We choose the sliding mode gains as $k = 4$ and $q = 0.1$.

We note that a large value of k can cause chattering and an appropriate value of q is chosen to speed up the time taken to reach the sliding manifold as well as to reduce the system chattering.

From Eq. (15), we can obtain $v(t)$ as

$$v(t) = -16.5e_1 - 9.75e_2 + 1.9e_4 + 0.025\,\text{sgn}(s). \tag{30}$$

Thus, the required sliding mode controller is obtained as

$$u = -\eta(x, y) + Bv \tag{31}$$

where $\eta(x, y), B$ and $v(t)$ are defined as in the equations (26), (28) and (30).

By Theorem 1, we obtain the following result.

Theorem 2. The identical hyperchaotic Lorenz systems (21) and (22) are globally and asymptotically anti-synchronized for all initial conditions with the sliding controller u defined by (31). ■

B. Numerical Results

For the numerical simulations, the fourth-order Runge-Kutta method with time-step $h = 10^{-6}$ is used to solve the hyperchaotic Lorenz chaotic systems (21) and (22) with the sliding controller u given by (31) using MATLAB.

In the hyperchaotic case, the parameter values are

$$a = 10, \quad r = 28, \quad b = 8/3 \quad \text{and} \quad d = 1.3.$$

The sliding mode gains are chosen as

$$k = 4 \quad \text{and} \quad q = 0.1.$$

The initial values of the master system (21) are taken as

$$x_1(0) = 26, \quad x_2(0) = 16, \quad x_3(0) = 32, \quad x_4(0) = 14$$

and the initial values of the slave system (22) are taken as

$$y_1(0) = 10, \quad y_2(0) = 25, \quad y_3(0) = 4, \quad y_4(0) = 40$$

Fig. 2 illustrates the anti-synchronization of the identical hyperchaotic Lorenz systems (21) and (22).

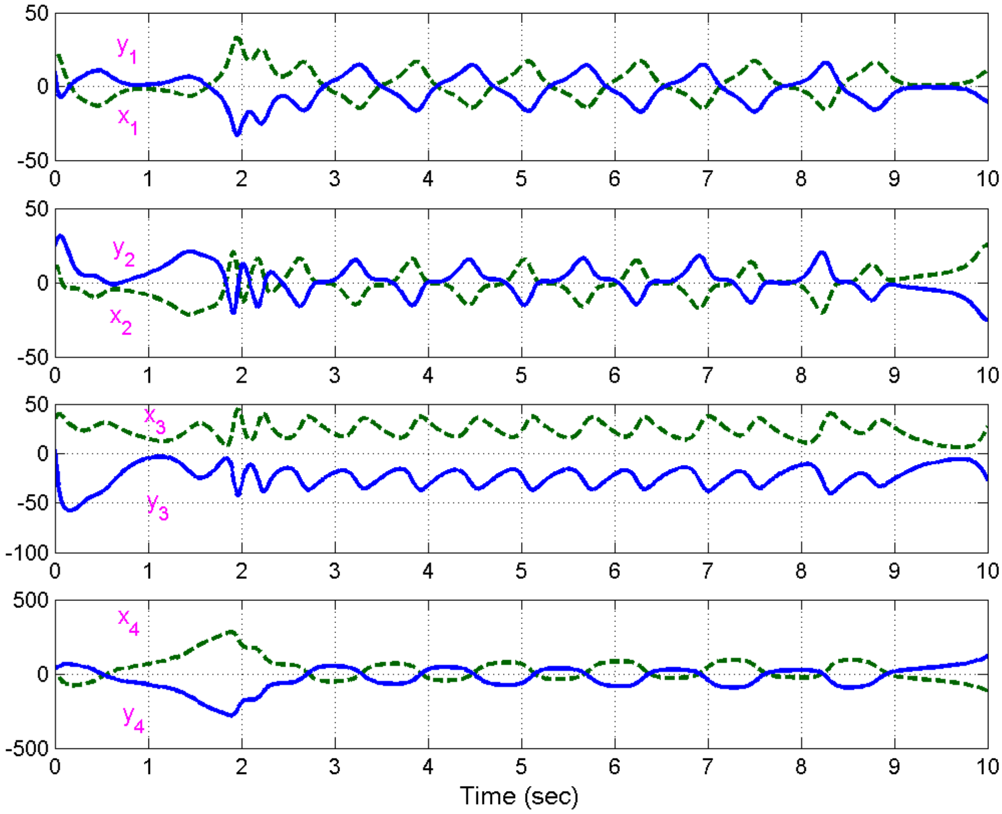

Figure 2. Anti-Synchronization of Identical Hyperchaotic Lorenz Systems

IV. Conclusions

In this paper, we have deployed sliding mode control to achieve global chaos anti-synchronization for the identical hyperchaotic Lorenz systems (2007). Our anti-synchronization results for the identical hyperchaotic Lorenz systems have been proved using Lyapunov stability theory. Since the Lyapunov exponents are not required for these calculations, the sliding control method is very effective and convenient to achieve global chaos anti-synchronization for the identical hyperchaotic Lorenz systems. Numerical simulations are also shown to illustrate the effectiveness of the anti-synchronization results derived in this paper using the sliding mode control.

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