Output Regulation of the Tigan System

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***Abstract*—This paper solves the problem of regulating the output of the Tigan system (2008), which is one of the recently discovered three-dimensional chaotic attractors. Tigan system has many interesting complex dynamical behaviours and it has potential applications in secure communications. In this paper, we construct explicit state feedback control laws to regulate the output of the Tigan system so as to track constant reference signals. The control laws are derived using the regulator equations of Byrnes and Isidori (1990), who have solved the output regulation of nonlinear systems involving neutrally stable exosystem dynamics. Numerical simulations are shown to validate the output regulation results derived in this paper.**

Keywords- nonlinear control systems; feedback stabilization; output regulation; chaos; Tigan system.

I. Introduction

Output regulation of nonlinear control systems is one of the very miscellaneous problems in nonlinear control theory. The output regulation problem is the problem of controlling a fixed linear or nonlinear plant in order to have its output tracking the reference signals produced by some external input generator (the *exosystem*).

For linear control systems, the output regulation problem was solved by Francis and Wonham ([1], 1975). For nonlinear control systems, the output regulation problem was solved by Byrnes and Isidori ([2], 1990) generalizing the internal model principle obtained by Francis and Wonham [1]. Byrnes and Isidori [2] made an important assumption in their work which demands that the exosystem dynamics generating the reference and disturbance signals is a *neutrally stable* system (Lyapunov stable in both forward and backward time). This class of exosystem signals includes the important special cases of constant reference signals as well as periodic reference signals. Using Centre Manifold Theory for flows [3], Byrnes and Isidori derived regulator equations, which completely characterize the solution of the output regulation problem of nonlinear control systems. Regulator equations play an important role in solving output regulation problems of various nonlinear control systems arising in practical applications in science and engineering.

The output regulation problem for linear and nonlinear control systems has been the focus of many studies in recent years [4-14]. In [4], Mahmoud and Khalil derived results on the asymptotic regulation of minimum phase nonlinear systems using output feedback. In [5], Fridman solved the output regulation problem for nonlinear control systems with delay. In [6-7], Chen and Huang obtained results on the global robust output regulation problem for lower triangular systems with unknown control direction. In [9], Immonen obtained results on the practical output regulation for bounded linear infinite-dimensional state space systems. In [10], Pavlov, Van de Wouw and Nijmeijer obtained results on the global nonlinear output regulation using convergence-based controller design. In [11], Xi and Ding, obtained results on the global adaptive output regulation of a class of nonlinear systems with nonlinear exosystems. In [12-14], Marconi and Isidori obtained results on the semi-global and global output regulation problems for minimum-phase nonlinear systems.

In this paper, we solve the output regulation problem for the Tigan system (2008) using the regulator equations [2] to derive the state feedback control laws for regulating the output of the Tigan system for the case of constant reference signals (*set-point signals*). Tigan system is one of the recent three-dimensional chaotic attractors derived by G. Tigan and D. Opris (2008). Tigan system has many interesting complex dynamical behaviours and it has potential applications in secure communications.

This paper is organized as follows. In Section II, we present a detailed review of the output regulation problem for nonlinear control systems and its solution (*regulator equations*) obtained by Byrnes and Isidori (1990). In Section III, we detail the main results obtained in this paper for the output regulation of the Tigan system (2009). In Section IV, we describe numerical simulations which illustrate the effectiveness of the regulating feedback laws derived in Section III for the Tigan system. In Section V, we summarize the main results obtained in this paper for the output regulation of the Tigan system.

II. Review of Output Regulation For Nonlinear Control Systems

In this section, we consider a multivariable nonlinear control system modelled by equations of the form

$$\dot{x} = f(x) + g(x)u + p(x)\omega \tag{1a}$$

$$\dot{\omega} = s(\omega) \tag{1b}$$

$$e = h(x) - q(\omega) \tag{2}$$

Here, the differential equation (1a) describes the plant dynamics with state x defined in a neighbourhood X of the origin of R^n and the input u takes values in R^m subject to the effect of a disturbance represented by the vector field $p(x)\omega.$ The differential equation (1b) describes an autonomous system, known as the exosystem, defined in a neighbourhood W of the origin of R^k, which models the class of disturbance and reference signals taken into consideration.

We also assume that all the constituent mappings of the system (1) and the error equation (2), *viz.* f, g, p, s, h and q are C^1 mappings vanishing at the origin, *i.e.*

$$f(0) = 0,\ g(0) = 0,\ p(0) = 0,\ s(0) = 0,\ h(0) = 0 \ \text{ and } \ q(0) = 0.$$

Thus, for $u = 0,$ the composite system (1) has an equilibrium state $(x, \omega) = (0, 0)$ with zero error (2).

A *state feedback controller* for the composite system (1) has the form

$$u = \alpha(x, \omega) \tag{3}$$

where α is a C^1 mapping defined on $X \times W$ such that $\alpha(0, 0) = 0.$ Upon substitution of the feedback control law (3) in the composite system (1), we get the closed-loop system given by

$$\begin{aligned} \dot{x} &= f(x) + g(x)\alpha(x, \omega) + p(x)\omega \\ \dot{\omega} &= s(\omega) \end{aligned} \tag{4}$$

The purpose of designing a state feedback controller (3) is to achieve both internal stability and output regulation. Internal stability means that when the input is disconnected from (4) [*i.e.* when $\omega = 0$], the closed-loop system (4) has an exponentially stable equilibrium at $x = 0.$ Output regulation means that for the closed-loop system (4), for all initial states $(x(0), \omega(0))$ sufficiently close to the origin, $e(t) \to 0$ asymptotically as $t \to \infty.$ Formally, we can summarize these requirements as follows.

State Feedback Regulator Problem:

Find, if possible, a state feedback control law $u = \alpha(x, \omega)$ such that

(OR1) [*Internal Stability*] The equilibrium $x = 0$ of the dynamics

$$\dot{x} = f(x) + g(x)\alpha(x, 0)$$

is locally exponentially stable.

(OR2) [*Output Regulation*] There exists a neighbourhood U of $(x, \omega) = (0, 0)$ such that for each initial condition $(x(0), \omega(0))$ in U, the solution $(x(t), \omega(t))$ of the closed-loop system (4) satisfies

$$\lim_{t \to \infty} \left[h(x(t)) - q(\omega(t)) \right] = 0. \quad \blacksquare$$

Byrnes and Isidori solved the state feedback regulator problem under the following assumptions:

(H1) The exosystem dynamics $\dot{\omega} = s(\omega)$ is neutrally stable at $\omega = 0,$ *i.e.* the system is Lyapunov stable in both forward and backward time at $\omega = 0.$

(H2) The pair $(f(x), g(x))$ has a stabilizable linear approximation at $x = 0$, *i.e.* if

$$A = \left[\frac{\partial f}{\partial x}\right]_{x=0} \quad \text{and} \quad B = \left[\frac{\partial g}{\partial x}\right]_{x=0},$$

then (A, B) is stabilizable, i.e. we can find a gain matrix K such that $A + BK$ is Hurwitz. ■

Theorem 1. [2] Under the assumptions (H1) and (H2), the state feedback regulator problem is solvable if and only if there exist C^1 mappings $x = \pi(\omega)$ with $\pi(0) = 0$ and $u = \varphi(\omega)$ with $\varphi(0) = 0$, both defined in a neighbourhood $W^0 \subset W$ of $\omega = 0$ such that the following equations (called the *regulator equations*) are satisfied:

(1) $\dfrac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\omega)) + g(\pi(\omega))\varphi(\omega) + p(\pi(\omega))\omega$

(2) $h(\pi(\omega)) - q(\omega) = 0$

When the regulator equations (1) and (2) are satisfied, a control law solving the feedback regulator problem is given by

$$u = \varphi(\omega) + K[x - \pi(\omega)] \tag{5}$$

where K is any gain matrix such that $A + BK$ is Hurwitz. ■

III. Output Regulation of the Tigan System

The Tigan system is a new three-dimensional chaotic attractor discovered by G. Tigan and D. Opris ([15], 2008) and described by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c - a)x_1 - a x_1 x_3 + u \\ \dot{x}_3 &= -b x_3 + x_1 x_2 \end{aligned} \tag{6}$$

where $a > 0,\ b > 0,\ c > 0$ are the parameters of the system and u is the control. The Tigan system (6) is chaotic when $u = 0$ and the parameter values are chosen as $a = 2.1,\ b = 0.6,\ c = 30.$ The chaotic portrait of the unforced Tigan system is shown in Figure 1.

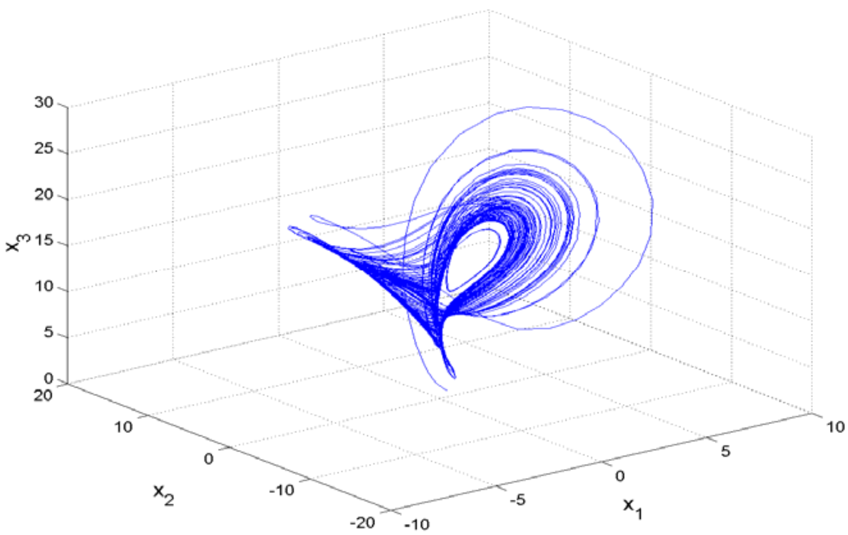

Figure 1. Chaotic State Portrait of the Tigan System

In this paper, we solve the problem of output regulation for the Tigan system (6) for tracking of the constant reference signals (*set-point signals*).

The constant reference signals are generated by the exosystem dynamics

$$\dot{\omega} = 0 \tag{7}$$

It is important to observe that the exosystem given by (7) is neutrally stable.

This follows simply because the differential equation (7) admits only constant solutions, *i.e.*

$$\omega(t) \equiv \omega(0) = \omega_0 \text{ for all } t \in R \tag{8}$$

Thus, the assumption (H1) of Theorem 1 holds trivially.

Linearizing the dynamics of the Tigan system (6) at the equilibrium $(x_1, x_2, x_3) = (0,0,0)$, we get

$$A = \begin{bmatrix} -a & a & 0 \\ c-a & 0 & 0 \\ 0 & 0 & -b \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Using Kalman's rank test for controllability [16], it can be easily seen that the pair (A, B) is not controllable. However, it can be easily shown that the pair (A, B) is stabilizable as follows.

The characteristic equation of the matrix $A + BK$ is obtained as

$$(\lambda + b)\left[\lambda^2 + \lambda(a - k_2) - a(k_1 + k_2 + c - a)\right] = 0 \tag{9}$$

where $K = [k_1 \quad k_2 \quad k_3]$.

Since $b > 0$, $\lambda = -b$ is a stable eigenvalue of $A + BK$ and the other two eigenvalues of $A + BK$ will be stable if and only if

$$a - k_2 > 0 \quad \text{and} \quad -a(k_1 + k_2 + c - a) > 0.$$

Since $a > 0$, the above conditions are equivalent to

$$k_2 < a \quad \text{and} \quad k_1 + k_2 + c < a. \qquad 10)$$

Since k_3 does not play any role in the above eigenvalue calculations, we can take $k_3 = 0$.

Thus, we shall assume that $K = [k_1 \quad k_2 \quad 0]$, where k_1 and k_2 are chosen so that the inequalities (10) are satisfied, *i.e.* such that $A + BK$ is Hurwitz.

This shows that (A, B) is stabilizable.

Hence, the assumption (H2) of Theorem 1 also holds.

Hence, we can apply Theorem 1 to completely solve the output regulation problem for the Tigan system for the tracking of constant reference signals (*set-point signals*).

Case (A): Constant Tracking Problem for x_1

Here, the tracking problem for the Tigan system is given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c-a)x_1 - ax_1x_3 + u \\ \dot{x}_3 &= -bx_3 + x_1x_2 \\ e &= x_1 - \omega \end{aligned} \tag{11}$$

The regulator equations for the system (11) are given by Theorem 1 as

$$\begin{aligned} a\left[\pi_2(\omega) - \pi_1(\omega)\right] &= 0 \\ (c-a)\pi_1(\omega) - a\pi_1(\omega)\pi_3(\omega) + \varphi(\omega) &= 0 \\ -b\pi_3(\omega) + \pi_1(\omega)\pi_2(\omega) &= 0 \\ \pi_1(\omega) - \omega &= 0 \end{aligned} \tag{12}$$

Solving the regulator equations (12), we obtain the unique solution

$$\begin{aligned} &\pi_1(\omega) = \omega,\ \pi_2(\omega) = \omega,\ \pi_3(\omega) = \frac{\omega^2}{b} \\ &\varphi(\omega) = (a-c)\omega + \frac{a\omega^3}{b} \end{aligned} \tag{13}$$

Using Theorem 1 and the solution (13) of the regulator equations (12), we obtain the following result which gives a state feedback control law solving the output regulation problem for the system (11).

Theorem 2. A state feedback control law solving the output regulation problem for the system (11) is given by

$$u = \varphi(\omega) + K\left[x - \pi(\omega)\right], \tag{14}$$

where $\varphi(\omega)$ and $\pi(\omega)$ are as given in Eq. (13) and the gain matrix

$$K = \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix}$$

is chosen so as to satisfy the inequalities (10). ■

Case (B): Constant Tracking Problem for x_2

Here, the tracking problem for the Tigan system is given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c-a)x_1 - ax_1x_3 + u \\ \dot{x}_3 &= -bx_3 + x_1x_2 \\ e &= x_2 - \omega \end{aligned} \tag{15}$$

The regulator equations for the system (15) are given by Theorem 1 as

$$\begin{aligned} a\left[\pi_2(\omega) - \pi_1(\omega)\right] &= 0 \\ (c-a)\pi_1(\omega) - a\pi_1(\omega)\pi_3(\omega) + \varphi(\omega) &= 0 \\ -b\pi_3(\omega) + \pi_1(\omega)\pi_2(\omega) &= 0 \\ \pi_2(\omega) - \omega &= 0 \end{aligned} \tag{16}$$

Solving the regulator equations (16), we obtain the unique solution

$$\pi_1(\omega) = \omega,\ \pi_2(\omega) = \omega,\ \pi_3(\omega) = \frac{\omega^2}{b}$$
$$\varphi(\omega) = (a-c)\omega + \frac{a\omega^3}{b} \tag{17}$$

Using Theorem 1 and the solution (17) of the regulator equations (16), we obtain the following result which gives a state feedback control law solving the output regulation problem for the system (15).

Theorem 3. A state feedback control law solving the output regulation problem for the system (15) is given by

$$u = \varphi(\omega) + K\left[x - \pi(\omega)\right], \tag{18}$$

where $\varphi(\omega)$ and $\pi(\omega)$ are as given in Eq. (17) and the gain matrix

$$K = \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix}$$

is chosen so as to satisfy the inequalities (10). ■

Case (C): Constant Tracking Problem for x_3

Here, the tracking problem for the Tigan system is given by

$$\begin{aligned} \dot{x}_1 &= a(x_2 - x_1) \\ \dot{x}_2 &= (c-a)x_1 - a x_1 x_3 + u \\ \dot{x}_3 &= -b x_3 + x_1 x_2 \\ e &= x_3 - \omega \end{aligned}$$

(19)

The regulator equations for the system (19) are given by Theorem 1 as

$$\begin{aligned} a\left[\pi_2(\omega) - \pi_1(\omega)\right] &= 0 \\ (c-a)\pi_1(\omega) - a\pi_1(\omega)\pi_3(\omega) + \varphi(\omega) &= 0 \\ -b\pi_3(\omega) + \pi_1(\omega)\pi_2(\omega) &= 0 \\ \pi_3(\omega) - \omega &= 0 \end{aligned}$$

(20)

Solving the regulator equations (20), we obtain the unique solution

$$\pi_1(\omega) = \sqrt{b\omega},\ \pi_2(\omega) = \sqrt{b\omega},\ \pi_3(\omega) = \omega$$
$$\varphi(\omega) = \left(a - c + a\omega\right)\sqrt{b\omega} \tag{21}$$

Using Theorem 1 and the solution (21) of the regulator equations (20), we obtain the following result which gives a state feedback control law solving the output regulation problem for the system (19).

Theorem 4. A state feedback control law solving the output regulation problem for the system (19) is given by

$$u = \varphi(\omega) + K\left[x - \pi(\omega)\right], \tag{22}$$

where $\varphi(\omega)$ and $\pi(\omega)$ are as given in Eq. (21) and the gain matrix

$$K = \begin{bmatrix} k_1 & k_2 & 0 \end{bmatrix}$$

is chosen so as to satisfy the inequalities (10). ■

IV. NUMERICAL RESULTS

For numerical simulation, we consider the chaotic case studied by Tigan and Opris (2008), *viz.*

$$a = 2.1, \quad b = 0.6 \quad \text{and} \quad c = 30.$$

We also consider the set-point control as $\omega_0 = 2.$

For achieving internal stability of the Tigan system, we must choose a gain matrix K such that $A + BK$ is Hurwitz. As shown in Section III, $\lambda = -b = -0.6$ is the uncontrollable, stable eigenvalue of the closed-loop system matrix $A + BK$. We choose the other two stable eigenvalues of $A + BK$ as $\{-2, -2\}$.

Thus, the desired characteristic equation of $A + BK$ is given by

$$(\lambda + b)(\lambda^2 + 4\lambda + 4) = 0 \tag{23}$$

In Section III, we showed that the characteristic equation of $A + BK$ is given by

$$(\lambda + b)\left[\lambda^2 + \lambda(a - k_2) - a(k_1 + k_2 + c - a)\right] = 0 \tag{24}$$

Equating (23) and (24), a simple calculation gives

$$K = [k_1 \quad k_2 \quad 0] = [-27.9048 \quad -1.9 \quad 0].$$

Case (A): Constant Tracking Problem for x_1

Suppose that $x(0) = (20, 10, 5)$ and $\omega_0 = 2.$

The simulation graph is depicted in Figure 2 from which it is clear that the state $x_1(t)$ tracks the constant signal $\omega = 2$ in 4 sec.

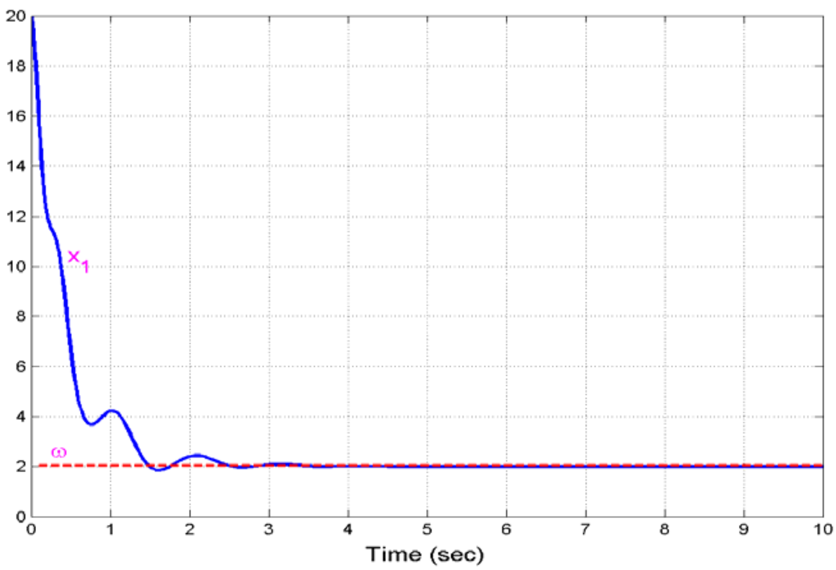

Figure 2. Constant Tracking Problem for x_1

Case (B): Constant Tracking Problem for x_2

Suppose that $x(0) = (8, 24, 6)$ and $\omega_0 = 2.$

The simulation graph is depicted in Figure 4 from which it is clear that the state $x_2(t)$ tracks the constant signal $\omega = 2$ in 4 sec.

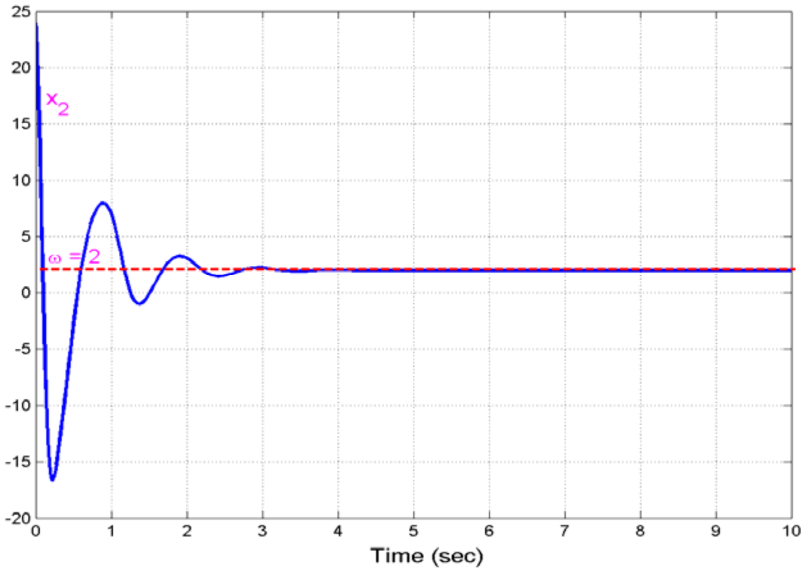

Figure 3. Constant Tracking Problem for x_2

Case (C): Constant Tracking Problem for x_3

Suppose that $x(0) = (6,12,15)$ and $\omega_0 = 2.$

The simulation graph is depicted in Figure 4 from which it is clear that the state $x_3(t)$ tracks the constant signal $\omega = 2$ in 4 sec.

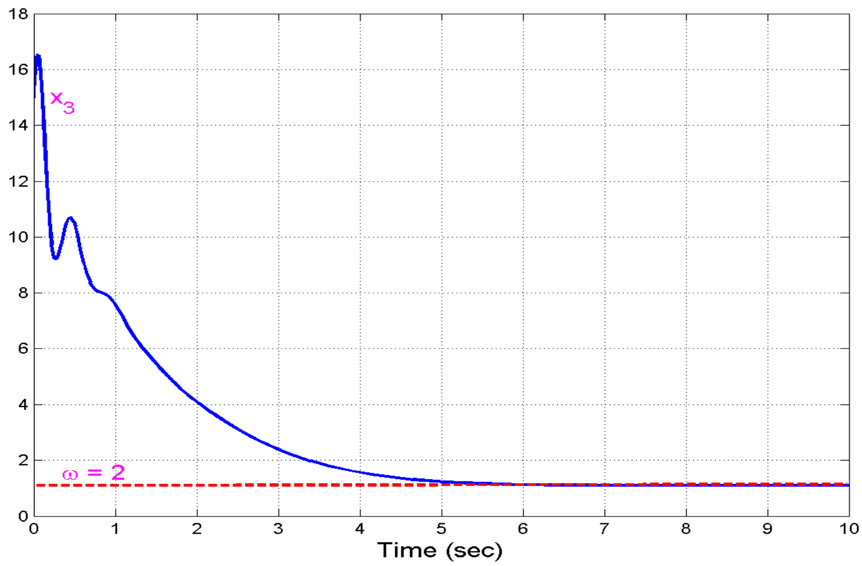

Figure 4. Constant Tracking Problem for x_3

V. Conclusions

In this paper, we studied in detail the output regulation for the Tigan system (2009) for the tracking of constant reference signals (set-point signals). Using the regulator equations of Byrnes and Isidori (1990), we derived the complete solution of the output regulation problem for Tigan system by deriving state feedback control laws so as to achieve constant tracking for the states of the Tigan system. Numerical simulations have been presented so as to validate and illustrate the effectiveness of the theoretical results obtained in this paper for the output regulation of the Tigan system.

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