AN EXTENDED METHOD FOR ORDER REDUCTION OF LARGE SCALE MULTI VARIABLE SYSTEMS

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Abstract

Abstract: In this paper, an effective procedure to determine the reduced order model of higher order linear time invariant dynamic multivariable systems is discussed. Numerator and denominator polynomials of reduced order model are obtained by redefining the time moments of the original high order system and the method is extended to multi variable systems. The proposed method has been verified using typical numerical examples.

Keywords: order reduction, eigen values, large scale systems, multivariable systems, modeling.

Introduction:

The rapid advancements in science and technology led to extreme research in large scale systems. As a result the overall mathematical complexity increases. The computational procedure becomes difficult with increase in dimension. Therefore high-order models are difficult to use for simulation, analysis or controller synthesis. So it is not only desirable but necessary to obtain satisfactory reduced order representation of such higher order models. Here the objective of the model reduction of high order complex systems is to obtain a Reduced Order Model(ROM) that retains and reflects the important characteristics of the original system as closely as possible.

Several methods are available in the literature for large-scale system modeling. Most of the methods based on the original continued fraction expansion technique [1, 2] fail to retain the stability of the original systems in the reduced order models. To overcome this major draw back, alternative methods have been suggested but the common limitation of such extension is that in some cases, they may generate models of order even higher than that of the original system [1-3].

Modal-Padé methods [4] use the concept of the dominant poles and matching the few initial time moments of the original systems. The major disadvantage of such methods is the difficulty in deciding the dominant poles of the original system, which should be retained in the reduced order models. Retaining the poles closest to the imaginary axis need not be always the best choice. Sinha et al, [5] used the clustering technique but the serious limitation of this method is the number and position of zeros of the original system sometimes decides the minimum order of the reduced order model. The major disadvantage of such methods is in deciding the clusters of poles hence cannot generate unique models.

Some methods based on Eigen Spectrum which is the cluster of poles of high order system considered to derive the approximant. Pal *et al* proposed [6] pole-clustering using Inverse Distance Criterion and time-moment matching. Vishwakarma and R.Prasad [7,8] modified the pole clustering by an iterative method. The difficulty with these methods is in selecting poles for the clusters. Mukherjee [9], suggested a method based on Eigen Spectrum Analysis. The Eigen Spectrum consists of all poles of the high order system. The poles of the reduced model are evenly spaced between the first and last poles. Parmar *et al* [10] proposed a mixed method using Eigen Spectrum Analysis with Factor division algorithm to determine the numerator of the reduced model with known denominator. Parmar *et al* [11] proposed another mixed method using Eigen Spectrum Analysis equation and Particle Swarm method to find the reduced model. The methods based on eigen spectrum analysis cannot be applied to high order systems having complex poles. Saraswathi et al[17] proposed a method retaining some of the properties of original system based on eigenspectrum. The method can be applied for systems having both real and complex eigenvalues unlike the other existing methods [9-12].

In the proposed reduced order method the Time moments and Markov parameters are redefined as function of Residues and poles for strictly proper rational transfer functions extended to repeated poles. Poles of the Reduced Order Model (ROM) are selected by considering the highest contribution of each pole in redefined Time Moments (RTMs) and lowest contribution in Redefined Markov Parameters (RMPs). The reduction method is extended to linear time invariant multivariable systems considering the transfer matrix of the system.

Proposed method: Let the original high order linear time invariant continuous system of n^{th} order be

$$\dot{x} = Ax + Bu$$

$$y = Cx$$

Where x is the state vector, u is the input, y is the output and A, B, C are constant matrices with appropriate dimensions.

The transfer matrix, $[G(s)] = C(sI - A)^{-1}B$

$$[G(s)] = \begin{bmatrix} g_{11}(s) & \cdots & g_{1n}(s) \\ \vdots & \vdots & \vdots \\ g_{n1}(s) & \cdots & g_{nn}(s) \end{bmatrix}$$

$$g_{ij}(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots \dots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots \dots + a_1 s + a_0}; \ m < n,\ i=1,2,\ldots n;\ j=1,2,\ldots n \qquad \ldots(1)$$

$$= \frac{N(s)}{D(s)}$$

where $D(s) = \prod_{i=1}^{n}(s + \lambda_i)$; n is the number of poles and $N(s) = \prod_{i=1}^{m}(s + \delta_i)$; m is the number of zeros, $\lambda_i = (\alpha_i \pm j\beta_i)$ and $\delta_i = (\varepsilon_i \pm j\gamma_i)$ of high order system $G(s)$. The poles and zeros may be real and/or complex. If they are complex, they occur in conjugate pairs.

Let the reduced order model of k^{th} order be defined as

$$[R_k(s)] = \begin{bmatrix} r_{11}(s) & \cdots & r_{1k}(s) \\ \vdots & \vdots & \vdots \\ r_{k1}(s) & \cdots & r_{kk}(s) \end{bmatrix}$$

$$r_{ij}(s) = \frac{p_r s^r + p_{r-1}s^{r-1} + \cdots \dots + p_1 s + p_0}{s^k + q_{k-1}s^{k-1} + q_{k-2}s^{k-2} + \cdots \dots + q_1 s + q_0}; \ k < n,\ i=1,2,\ldots k;\ j=1,2,\ldots k \qquad \ldots(2)$$

$$= \frac{N_k(s)}{D_k(s)}$$

where $D_k(s) = \prod_{i=1}^{k}(s + \hat{\lambda}_i)$; k is the number of poles and $N(s) = \prod_{i=1}^{r}(s + \hat{\delta}_i)$; $r \le m$, r is the number of zeros, $\hat{\lambda}_i = (\hat{\alpha}_i \pm j\hat{\beta})$ and $\hat{\delta}_i = (\hat{\varepsilon}_i \pm j\hat{\gamma}_i)$ of reduced order model $R_k(s)$. In the reduced model poles and zeros may be real and/or complex. If they are complex, they occur in conjugate pairs as mentioned for original system.
We know that the power series expansion of $G(s)$ about $s = 0$ is

$$g_{ij}(s) = C_0 + C_1 s + C_2 s^2 + \cdots = \sum_{i=0}^{\infty} C_i s^i \qquad \ldots(3)$$

Where $C_i = \frac{(-1)^i}{i!} RTM_i$; $i = 0,1,2,3, \ldots$..

The expansion of $G(s)$ about $s = \infty$ is

$$g_{ij}(s) = \mu_0 s^{-(n-m)} + \mu_1 s^{-(n-m+1)} + \mu_2 s^{-(n-m+2)} + \cdots = \sum_{i=0}^{\infty} \mu_i s^{-(n-m+i)} \qquad \ldots(4)$$

Where $\mu_i = RMP_i$; i= 0,1, 2, 3, …..

i) Considering the original high order system $G(s)$ with distinct poles
Define the expressions for redefined time moments (RTMs) as

$$RTM_i = \sum_{j=1}^{n} x_{ij};\ i = 0,1,2,\ldots \qquad \ldots(5)$$

where $x_{ij} = i!\,\frac{P_j}{\lambda_j^{(i+1)}};\ i = 0,1,2,\ldots$

Define Redefined Markov Parameters (RMPs) as

$$RMP_i = \sum_{j=1}^{n} y_{ij} \qquad \ldots(6)$$

where $y_{ij} = P_j \lambda_j^i;\ i = 0,1,2,\ldots$ and P_j are residues.
ii)If $g_{ij}(s)$ is having ‘r’ repeated poles

$$g_{ij}(s) = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots \dots + b_1 s + b_0}{(s+\lambda_1)^r (s+\lambda_{r+1}) \dots (s+\lambda_n)}\ ;\ m < n$$

Define the expressions for redefined time moments (RTMs) as

$$RTM_i = i!\left[\frac{P_1}{\lambda_1^{(i+1)}} + [ET_{ix}] + \sum_{k=r+1}^{n}\frac{P_k}{\lambda_k^{(i+1)}}\right];\ i = 0,1,2..\qquad \ldots(7)$$

where $ET_{ix} = \left[\sum_{x=1}^{r-1}\frac{P_{(x+1)}}{\lambda_1^{(i+x+1)}}\left\{\prod_{j=1}^{x}\frac{(-1)^j(i+j)}{x!}\right\}\right]$ and P_j are residues.

Define Redefined Markov Parameters (RMPs) as

$$RMP_i = \left[P_1\lambda_1^i + [EM_{ix}] + \sum_{k=r+1}^{n}P_k\lambda_k^i\right];\ i = 0,1,2..\qquad \ldots(8)$$

where $EM_{ix} = \left[\sum_{x=1}^{r-1}P_{(x+1)}\lambda_1^{(i-x)}\left\{\prod_{j=1}^{x}\frac{(i-j+1)}{x!}\right\}\right]$; $E_{ix} = 0$ if $i \le x$ and P_j are residues.

The denominator polynomial $D_k(s)$ of the k^{th} order reduced model is obtained by selecting poles with the highest contribution in RTMs and lowest contribution in RMPs according to their contribution weight age as shown in Table I.

Table I Contributions of individual poles

Parameters	λ_1	λ_2	λ_3	…	λ_j	…	λ_n	Sum
RTMs	x_{i1}	x_{i2}	x_{i3}	…	x_{ij}	…	x_{in}	RTM_i
RMPs	y_{i1}	y_{i2}	y_{i3}	…	y_{ij}	…	y_{in}	RMP_i

Where x_{ij} is the contribution of pole λ_j in RTM_i and y_{ij} is the contribution of pole λ_j in RMP_i.
The numerator polynomial, $N_k(s)$ of the k^{th} order reduced model is obtained by retaining the first few initial RTMs and RMPs of the original system as follows:

$$N_k(s) = \sum_{i=0}^{r_1}p_i s^i + \sum_{i=1}^{r_2}p_{(r_1+i)}s^{(r_1+i)};\ r = r_1 + r_2,\ r \le m \text{ and } r_1 \ge 1.\qquad \ldots(9)$$

where $p_i = \sum_{i=0}^{j} q_i c_{(j-i)}$; $j = 0, 1, 2, 3,\ldots, r_1$; r_1 is number of RTMs

and $p_{(r-j)} = \sum_{i=0}^{j} q_{(r-i)}\mu_{(j-i)}$; $j = 0, 1, 2, 3,\ldots, r_2$; r_2 is number of RMPs

Example : Consider an example of two input and two output Multivariable system [13] given by a transfer matrix $[G(s)]$:

$$[G(s)] = \begin{bmatrix} \frac{2(s+5)}{(s+1)(s+10)} & \frac{(s+4)}{(s+2)(s+5)} \\ \frac{(s+10)}{(s+1)(s+20)} & \frac{(s+6)}{(s+2)(s+3)} \end{bmatrix}$$

Let
$$[G(s)] = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix}$$
$$= \frac{1}{D_6(s)}\begin{bmatrix} \alpha_{11}(s) & \alpha_{12}(s) \\ \alpha_{21}(s) & \alpha_{22}(s) \end{bmatrix}$$

Where

$$D_6(s) = s^6 + 41s^5 + 571s^4 + 3491s^3 + 10060s^2 + 13100s + 6000$$
$$\alpha_{11}(s) = 2(s^5 + 35s^4 + 381s^3 + 1805s^2 + 3850s + 3000)$$
$$\alpha_{12}(s) = s^5 + 38s^4 + 459s^3 + 2182s^2 + 4160s + 2400$$
$$\alpha_{21}(s) = s^5 + 30s^4 + 331s^3 + 1650s^2 + 3700s + 3000$$
$$\alpha_{22}(s) = s^5 + 42s^4 + 601s^3 + 3660s^2 + 9100s + 6000$$

The poles of the system are $-\lambda_1 = -1$, $\lambda_2 = -2$, $-\lambda_3 = -3$, $-\lambda_4 = -5$, $-\lambda_5 = -10$, $-\lambda_6 = -20$. The contributions of individual poles in RTMs and RMPs are derived using equations (5) and (6) for $g_{11}(s)$, $g_{12}(s)$, $g_{21}(s)$ and $g_{22}(s)$ are tabulated in Table II and Table III. Poles having highest contribution in RTMs and lowest contribution in RMPs as per their absolute values and their contribution weight age in all elements of transfer matrix are considered for the second order reduced model. The poles are $\lambda_1 = -1$ and $\lambda_2 = -2$.

Table II Contribution of poles in RTMs and RMPs of HOS

Contribution of poles in RTMs and RMPs in $g_{11}(s)$							Contribution of poles in RTMs and RMPs in $g_{12}(s)$						
	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
RTM_0 = 1.0	0.89	0	0	0	0.11	0	RTM_0= 0.40	0	0.33	0	0.07	0	0
RTM_1 = 0.8	0.89	0	0	0	0.01	0	RTM_1= 0.18	0	0.17	0	0.01	0	0
RMP_0 = 2.0	0.89	0	0	0	1.11	0	RMP_0 = 1.00	0	0.67	0	0.33	0	0
RMP_1 = -12.0	-0.89	0	0	0	-11.11	0	RMP_1= -3.00	0	-1.33	0	-1.67	0	0

Table III Contribution of poles in RTMs and RMPs of HOS

Contribution of poles in RTMs and RMPs in $g_{21}(s)$							Contribution of poles in RTMs and RMPs in $g_{22}(s)$						
	λ_1	λ_2	λ_3	λ_4	λ_5	λ_6		λ_1	λ_2	λ_3	λ_4	λ_5	λ_6
RTM_0 = 0.50	0.47	0	0	0	0	0.03	RTM_0 = 1.0	0	2.0	-1.0	0	0	0
RTM_1 = 0.475	0.47	0	0	0	0	0.005	RTM_1 = 0.67	0	1.0	-0.3	0	0	0
RMP_0 = 1.00	0.47	0	0	0	0	0.53	RMP_0 = 1.00	0	4.0	-3.0	0	0	0
RMP_1 = -11.0	-0.47	0	0	0	0	-10.53	RMP_1 = 1.00	0	-8.0	9.0	0	0	0

Let the second order model reduced model be $[R_2(s)]$

$$[R_2(s)] = \begin{bmatrix} r_{11}(s) & r_{12}(s) \\ r_{21}(s) & r_{22}(s) \end{bmatrix}$$

$$= \frac{1}{D_2(s)} \begin{bmatrix} \beta_{11}(s) & \beta_{12}(s) \\ \beta_{21}(s) & \beta_{22}(s) \end{bmatrix}$$

The denominator polynomial of the second order reduced model $[R_2(s)]$ is

$$D_2(s) = (s+1)(s+2)$$
$$= s^2 + 3s + 3$$

Four numerator polynomials $\beta_{11}(s)$, $\beta_{12}(s)$, $\beta_{21}(s)$ and $\beta_{22}(s)$ of the reduced order model, $[R_2(s)]$ are to be determined separately using RTMs and RMPs from Tables II and III. Two RTMs are considered here for numerator polynomials as per the fulfillment of contributions of the eigenvalues. Numerator Polynomial of second order reduced model $\beta_{11}(s)$ of Multivariable system is obtained using equation (9) by matching first two RTMs of $g_{11}(s)$ of the original system from Table II.

$$\beta_{11}(s) = 1.2s + 2$$

The transfer function $r_{11}(s)$ is

$$r_{11}(s) = \frac{1.2s+2}{s^2+3s+2}$$

Numerator Polynomial of second order reduced model $\beta_{12}(s)$ is obtained using equation (9) by matching first two RTMs of the original system from Table II.

$$\beta_{12}(s) = 0.84s + 0.8$$

The transfer function $r_{12}(s)$ is

$$r_{12}(s) = \frac{0.84s+0.8}{s^2+3s+2}$$

Numerator Polynomial of second order reduced model $\beta_{21}(s)$ is obtained using equation (9) by matching first two RTMs of the original system from Table III.

$$\beta_{21}(s) = 0.55s + 1$$

The transfer function $r_{21}(s)$ is

$$r_{21}(s) = \frac{0.55s+1}{s^2+3s+2}$$

Numerator Polynomial of second order reduced model $\beta_{22}(s)$ is obtained using equation (9) by matching first two RTMs of the original system from Table III.

$$\beta_{11}(s) = 1.667s + 2$$

The transfer function $n_{22}(s)$ is

$$n_{22}(s) = \frac{1.667s+2}{s^2+3s+2}$$

The corresponding Transfer Matrix is

$$[R_2(s)] = \frac{1}{s^2+3s+2}\begin{bmatrix} 1.2s+2 & 0.84s+0.8 \\ 0.55s+1 & 1.667s+2 \end{bmatrix}$$

The second order transfer Matrix by [S. Mukherjee [9] is

$$[R_2^M(s)] = \frac{1}{s^2+13.667s+8.471}\begin{bmatrix} 3.29s^2+23.61s+8.471 & 1.76s^2+10.10s+3.388 \\ 1.48s^2+11.75s+4.235 & 2.27s^2+25.05s+8.471 \end{bmatrix}$$

The second order transfer Matrix (Continued Fraction Expansion) by R.Prasad *et al* [14] is

$$[R_2^{PR}(s)] = \frac{1}{s^2+6.81s+8.946}\begin{bmatrix} 1.17s+8.946 & 0.82s+3.578 \\ -5.56s+4.479 & 2.23s+8.951 \end{bmatrix}$$

The second order transfer Matrix (Polynomial Differentiation) by R.Prasad *et al* [15] is

$$[R_2^P(s)] = \frac{1}{s^2+3s+2}\begin{bmatrix} 1.18s+2 & -0.29s+0.8 \\ 0.43s+1 & 1.27s+2 \end{bmatrix}$$

The second order transfer Matrix(Modal-Padé mixed method) by R.Prasad *et al*[16] is

$$[R_2^{RP}(s)] = \frac{1}{s^2+13.667s+38.067}\begin{bmatrix} 2s+38.067 & s+15.227 \\ s+19.033 & s+38.067 \end{bmatrix}$$

The second order transfer Matrix by Viswakarma *et al* [8] is

$$[R_2^V(s)] = \frac{1}{s^2+4.337s+3.651}\begin{bmatrix} 1.182s+3.651 & 1.047s+1.46 \\ 0.498s+1.825 & 1.691s+3.651 \end{bmatrix}$$

Comparison of step responses of proposed second order model are shown in Fig. 1, 2, 3 and 4 for the four combinations of inputs and outputs. Step responses of proposed models are compared with models of five other methods along with original system in Fig. 5, 6, 7 and 8. The outputs of step responses at 2 seconds are tabulated in Table IV. The step responses of the proposed reduction method are following the step responses of original high order system very closely. The step responses by other methods are reaching steady state at some point of time but the best is always the one which closely follows the original step response for all *t*.

Table IV Comparison of reduced models at 2 seconds

At 2 Secs.	Original	Proposed	Mukherjee [9]	Prasad *et al* [14]	R.Prasad *et al* [15]	Prasad *et al* [16]	Viswakarma *et al* [8]
Ip(1) & op(1)	0.886	0.884	1.19	0.886	0.89	0.974	0.901
Ip(2) & op(1)	0.394	0.39	0.483	0.266	0.4	0.392	0.387
Ip(1) & op(2)	0.437	0.437	0.595	0.425	0.5	0.442	0.445
Ip(2) & op(2)	0.96	0.931	1.23	0.897	1.0	0.983	0.925

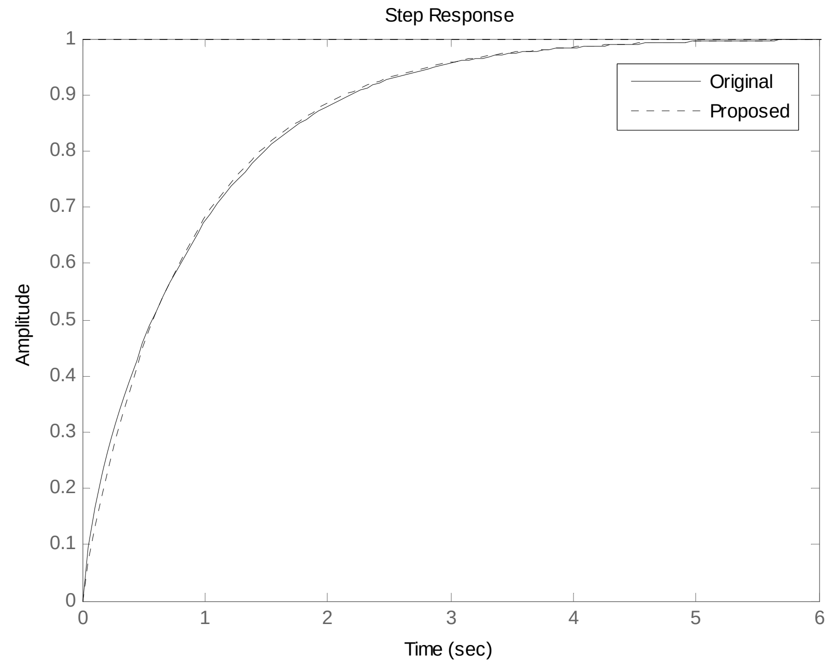

Fig. 1 Comparison of step responses from input(1) to output (1)

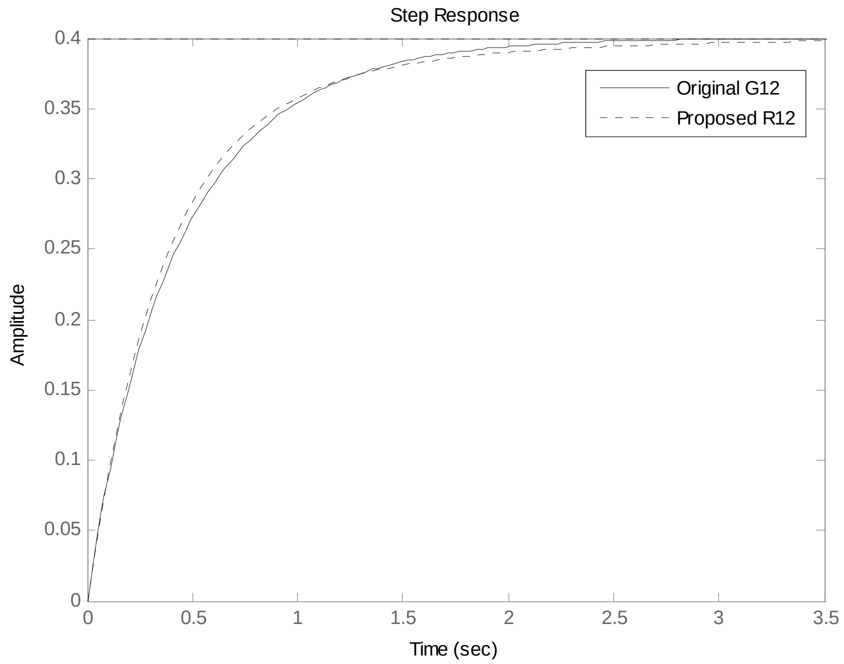

Fig. 2 Comparison of step responses from input(2) to output (1)

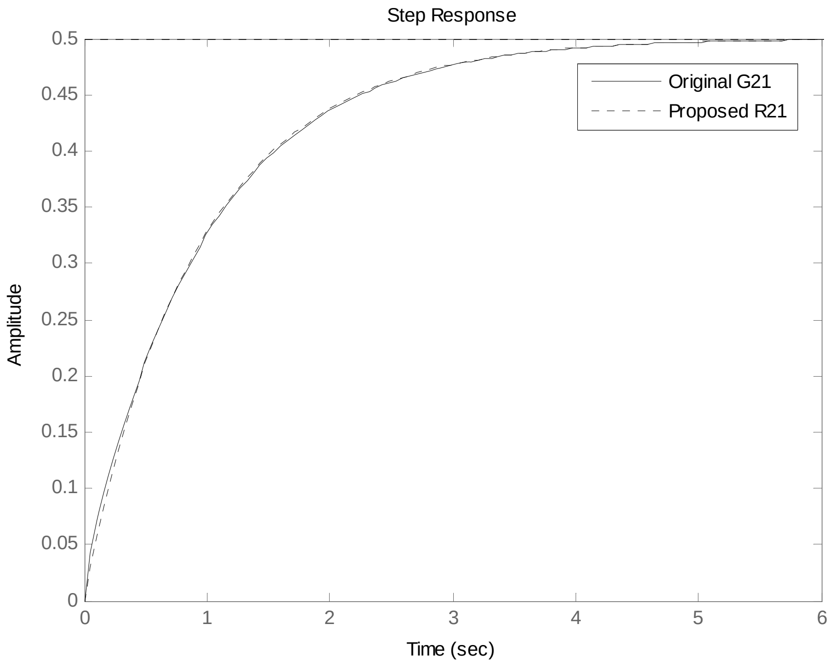

Fig. 3 Comparison of step responses from input(1) to output (2)

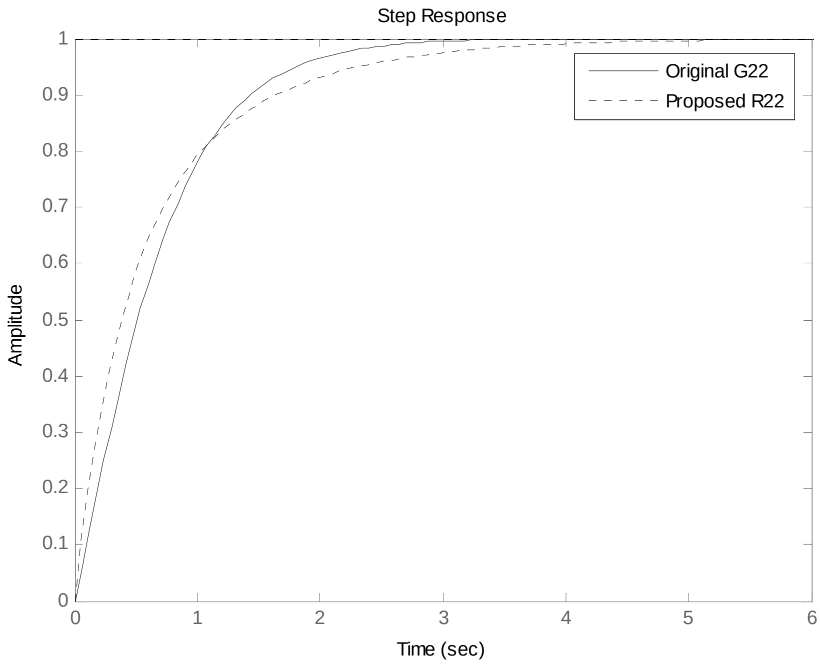

Fig. 4 Comparison of step responses from input(2) to output (2)

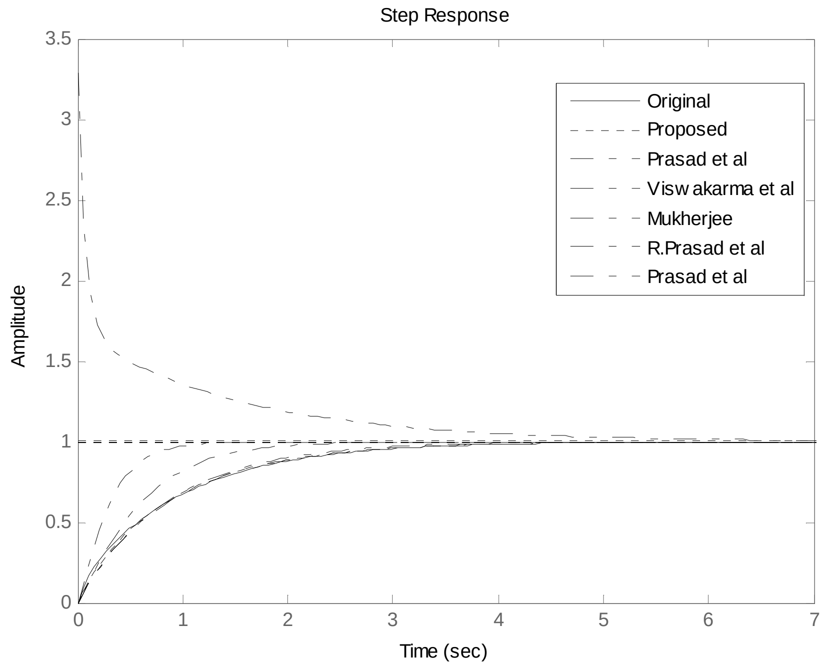

Fig. 5 Comparison of step responses from input(1) to output (1)

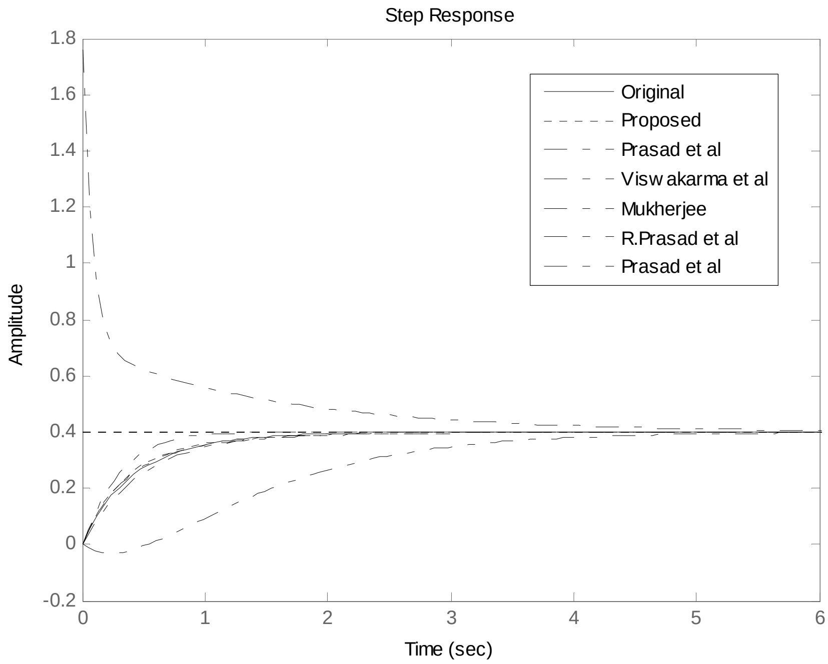

Fig. 6 Comparison of step responses from input(2) to output (1)

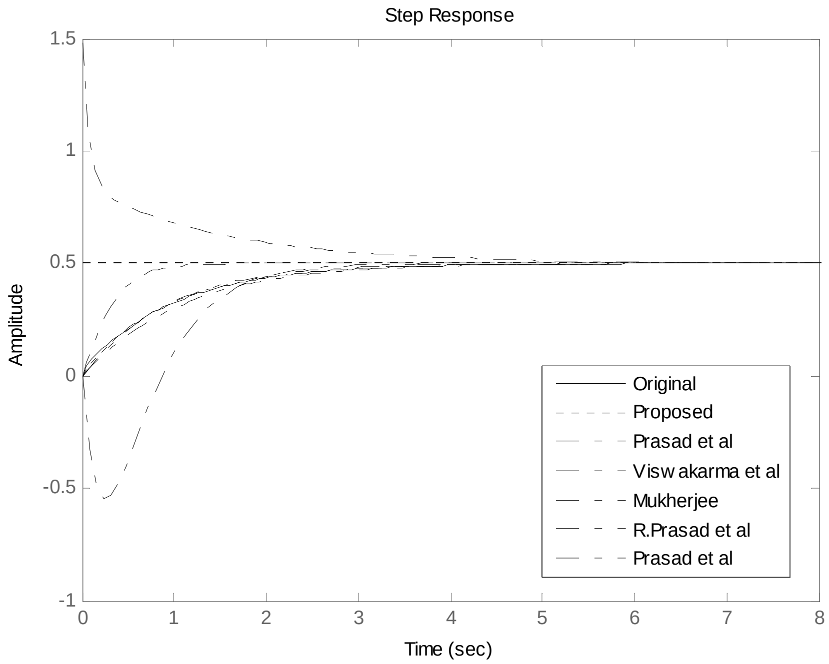

Fig. 7 Comparison of step responses from input(1) to output (2)

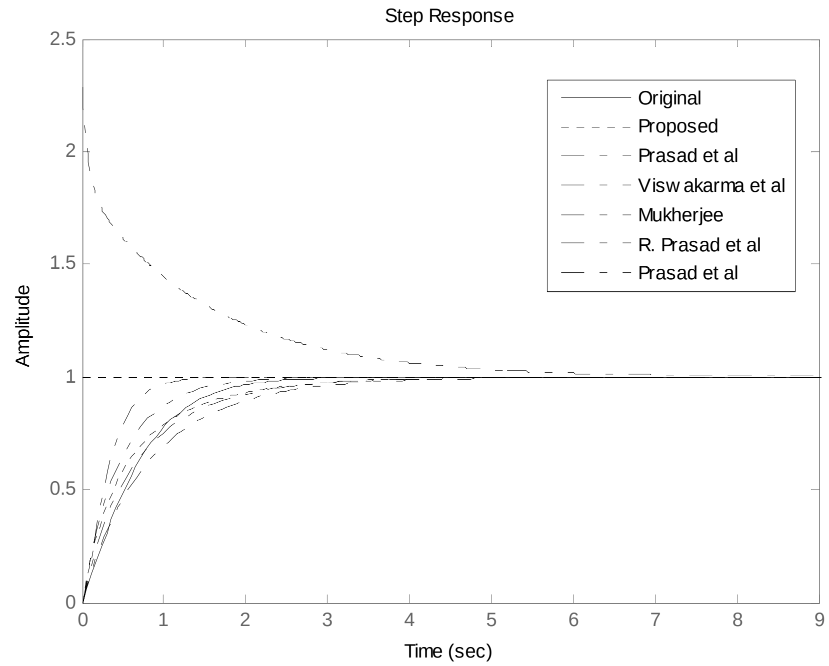

Fig.8 Comparison of step responses from input(2) to output (2)

Conclusions and Future Research: In this paper, an effective procedure to determine the reduced order model of higher order linear time invariant dynamic multivariable systems is presented.

Numerator and denominator polynomials of reduced order model are obtained by redefining the time moments of the original high order system. The stability of the original system is preserved in the reduced order model as the poles are taken from the original system. The method produces a good approximation when compared with other methods. The method is applied for real and complex poles and the work is in progress to make it generalize for discrete and interval systems.

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