Exponential Observers for Lotka-Volterra Systems

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***Abstract*—This paper solves the exponential observer design problem for Lotka-Volterra systems. Explicitly, Sundarapandian's theorem (2002) for observer design for exponential observer design is used to solve the nonlinear observer design problem for 2-species, 3-species and 4-species Lotka-Volterra systems. Numerical examples are provided to illustrate the effectiveness of the proposed exponential observer design for the Lotka-Volterra systems.**

Keywords-Exponential observers; Lotka-Volterra systems; nonlinear observers; observability.

I. INTRODUCTION

The nonlinear observer problem is one of the central problems in the control systems literature. In the control systems design, it is often necessary to construct estimates of state variables, which are not available for direct measurement. In such cases, the state vector of the control system can be approximately reconstructed by building an observer which is driven by the available outputs and inputs of the original control system. Local observer design for nonlinear control systems is one of the central problems in the control systems literature.

The problem of designing observers for linear control systems was first introduced by Luenberger ([1], 1966) and that for nonlinear control systems was proposed by Thau ([2], 1973). Over the past three decades, significant attention has been paid in the control systems literature to the construction of observers for nonlinear control systems.

A necessary condition for the existence of an exponential observer for nonlinear control systems was obtained by Xia and Gao ([3], 1988). Explicitly, in [3], Xia and Gao showed that an exponential observer exists for a nonlinear system only if the linearization of the nonlinear system is detectable.

On the other hand, sufficient conditions for nonlinear observers have been obtained in the control systems literature from an impressive variety of points of view. Kou, Elliott and Tarn ([4], 1975) obtained conditions for the existence of exponential observers using Lyapunov-like method. In [5-10], suitable coordinate transformations were found under which a nonlinear control system is transferred into a canonical form, where the observer design is carried out. In [11], Kazantzis and Kravaris obtained results on nonlinear observer design using Lyapunov auxiliary theorem. In [12-13], Tsinias derived sufficient Lyapunov-like conditions for the existence of asymptotic observers for nonlinear systems. A harmonic analysis approach was proposed by Celle *et. al.* [14] for the synthesis of nonlinear observers.

Necessary and sufficient conditions for the existence of local exponential observers for nonlinear control systems were obtained using geometric techniques by Sundarapandian ([15], 2002). Krener and Kang ([16], 2003) introduced a new method for the design of observers for nonlinear systems using backstepping.

An important interactive model of nonlinear systems is the two species model discovered independently by the Italian mathematician Vito Volterra ([17], 1926) and the American biophysicist ([18], 1925). This important model of simultaneous differential equations paved the way to multispecies population models and a formal study of food chains and ecosystems ([19], 2002).

Recently, there has been significant interest in the applications of mathematical systems theory to population biology systems [20-23]. A survey paper by Varga ([20], 2008) reviews the research done in this area. Scarelli and Varga ([21], 2002) obtained results for the controllability of selection-mutation models. Lopez, Gamez and Varga ([22], 2005) obtained results on the observability and controllability in selection-mutation models. Lopez, Gamez and Molnar ([23], 2007) obtained results on the observability and observers in a food web.

This paper is organized as follows. In Section II, we review the definition of nonlinear observers and observability and the results of observers and observability for nonlinear systems. In Section III, we derive new results on the design of exponential observers for two species Lotka-Volterra systems. In Section IV, we derive new results on the design of exponential observers for three species Lotka-Volterra systems. In Section V, we derive new results on the design of exponential observers for four species Lotka-Volterra systems. Finally, Section VI provides the conclusions of this paper.

II. REVIEW OF OBSERVABILITY AND OBSERVERS FOR NONLINEAR SYSTEMS

By the concept of a state observer, it is meant that from the observation of certain states of the system considered as outputs or indicators, it is desired to estimate the state of the whole system as a function of time.

Consider the nonlinear system described by

$$\dot{x} = f(x) \tag{1a}$$

$$y = h(x) \tag{1b}$$

Where $x \in R^n$ is the state and $y \in R^p$ the output. It is assumed that $f : R^n \to R^n$, $h : R^n \to R^p$ are C^1 mappings and for some $x^* \in R^n$, the following hold:

$$f(x^*) = 0, \quad h(x^*) = 0.$$

Note that the solutions x^* of the equation $f(x) = 0$ are called equilibrium points of the dynamics (1a).

Definition 1. The nonlinear system (1) is called *locally observable* at the equilibrium x^* over a given time interval $[0, T]$, if there exists $\varepsilon > 0$ such that for any two different solutions x and ξ of the system (1a) with

$$\left| x(t) - x^* \right| < \varepsilon \ \text{ and } \ \left| \xi(t) - x^* \right| < \varepsilon \ \text{ for } t \in [0, T],$$

the observed functions $h \circ x$ and $h \circ \xi$ are different, *i.e.* there exists at least one value $\tau \in [0, T]$ such that

$$(h \circ x)(\tau) \neq (h \circ \xi)(\tau). \quad \blacksquare$$

For the formulation of a sufficient condition for local observability of the nonlinear system (1), consider the linearization of (1) at the equilibrium $x = x^*$ given by

$$\dot{x} = Ax \tag{2a}$$

$$y = Cx \tag{2b}$$

where

$$A = \left[\frac{\partial f}{\partial x} \right]_{x = x^*} \ \text{ and } \ C = \left[\frac{\partial h}{\partial x} \right]_{x = x^*}.$$

Theorem 1 (Lee and Markus, 1971)

If the observability matrix for the linear system (2) given by

$$\boldsymbol{O}(C, A) = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

has rank n, then the nonlinear system (1) is locally observable at $x = x^*$. $\blacksquare$

Definition 3 (Sundarapandian, 2002)

A C^1 dynamical system described by

$$\dot{z} = g(z, y), \qquad \left(z \in R^n \right) \tag{3}$$

Is called a local asymptotic (respectively, exponential) observer for the nonlinear system (1) if the composite system (1) and (3) satisfies the following two requirements:

(O1) If $z(0) = x(0)$, then $z(t) = x(t)$ for all $t \geq 0$.

(O2) There exists a neighbourhood V of the equilibrium $x = x^*$ of R^n such that for all $z(0), x(0) \in V$, the estimation error $e(t) = z(t) - x(t)$ decays asymptotically (respectively, exponentially) to zero. ■

Theorem 2 (Sundarapandian, 2002)

Suppose that the nonlinear system (1) is Lyapunov stable at the equilibrium $x = x^*$ and that there exists a matrix K such that $A - KC$ is Hurwitz. Then the nonlinear system defined by

$$\dot{z} = f(z) + K[y - h(z)] \tag{4}$$

is a local exponential observer for the nonlinear system (1). ■

Remark 1. If the estimation error is defined as $e = z - x$, then the estimation error is governed by the dynamics

$$\dot{e} = f(x+e) - f(x) - K\left[h(x+e) - h(x)\right] \tag{5}$$

Linearizing the error dynamics (5) at $x = x^*$ yields the system

$$\dot{e} = Ee, \text{ where } E = A - KC. \tag{6}$$

If (C, A) is observable, i.e. if the observability matrix $\mathcal{O}(C, A)$ has full rank, then the eigenvalues of $E = A - KC$ can be arbitrarily assigned in the complex plane. Since the linearization of the error dynamics (5) is governed by the system matrix E, it follows that when (C, A) is observable, then a local exponential observer of the form (4) can be always found so that the transient response of the error decays quickly with any desired speed of convergence. ■

III. Exponential Observers for Two Species Lotka-Volterra Systems

In the 1920s, the Italian mathematician, Vito Volterra ([17], 1926) proposed a differential equation model to describe the population dynamics of two interacting species, a *predator* and its *prey*. With this model, Volterra hoped to explain the observed increase in predator fish and corresponding decrease in the prey fish in the Adriatic Sea during the World War I. Such mathematical models have long proven useful in describing how populations of various species evolve over time.

Independently, the American biophysicist, Alfred Lotka ([18], 1925) discovered the very same differential equation model to describe a hypothetical chemical reaction in which the chemical concentrations oscillate. Collectively, the interacting two species population dynamics model is referred to as *Lotka-Volterra system*.

The two species Lotka-Volterra system consists of the following system of differential equations

$$\begin{aligned} \dot{x}_1 &= ax_1 - bx_1x_2 \\ \dot{x}_2 &= -cx_2 + dx_1x_2 \end{aligned} \tag{7}$$

where $x_2(t)$ and $x_1(t)$ represent, respectively, the predator population and the prey population as functions of time. In the model (7), the parameters $a, b, c, d > 0$ have the following interpretation:

(i) a represents the natural growth rate of the prey in the absence of predators.

(ii) b represents the effect of predation on the prey.

(iii) c represents the natural death rate of the predator in the absence of prey.

(iv) d represents the efficiency and propagation rate of the predator in the presence of prey.

The equilibrium points of the Lotka-Volterra system (7) are obtained by setting $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$, and solving the resulting nonlinear equations for x_1 and x_2:

$$\begin{aligned} x_1(a - bx_2) &= 0 \\ x_2(-c + dx_1) &= 0 \end{aligned} \tag{8}$$

By solving the system (8), we obtain the equilibrium points, $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $x^* = \begin{bmatrix} c/d \\ a/b \end{bmatrix}$.

The linearization matrix of the nonlinear system (7) at the origin is given by

$$A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$$

which has a positive eigenvalue $\lambda_1 = a$ and a negative eigenvalue $\lambda_2 = -c$. This shows that the equilibrium $x = \mathbf{0}$ of the Lotka-Volterra system (7) is a saddle point, which is unstable.

On the other hand, the linearization matrix of the nonlinear system (7) at the equilibrium $x = x^*$ is given by

$$A = \frac{\partial f}{\partial x}(x^*) = \begin{bmatrix} 0 & -bc/d \\ ad/b & 0 \end{bmatrix} \tag{9}$$

which has purely imaginary eigenvalues $\lambda = \pm j\sqrt{ac}$. This is called a *critical case* in stability analysis.

More insight into the stability nature of the equilibrium x^* is obtained by eliminating t between the two differential equations in the Lotka-Volterra system (7) and integrating the resulting separable ODE:

$$\frac{dx_2}{dx_1} = \frac{\dot{x}_2}{\dot{x}_1} = \frac{x_2(-c + dx_1)}{x_1(a - bx_2)} \tag{10}$$

Integration of (10) yields the family of state orbits

$$a \ln x_2 - bx_2 + c \ln x_1 - dx_1 = k, \tag{11}$$

where k is the constant of integration.

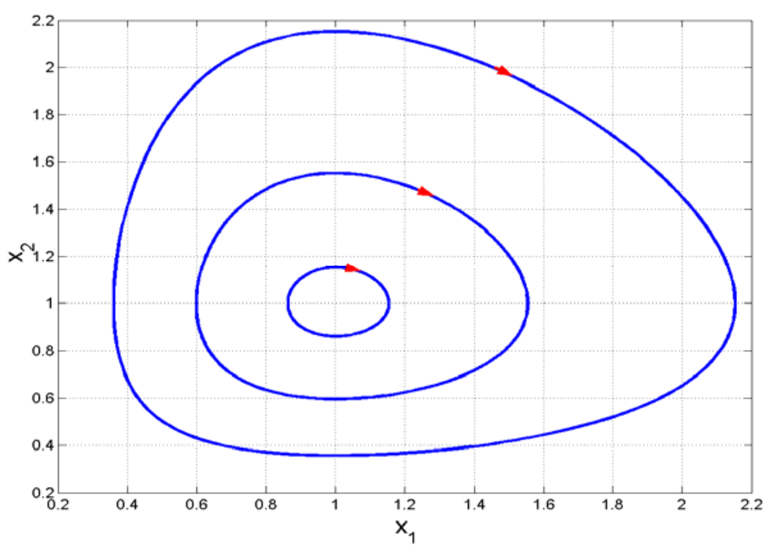

Figure 1. State Orbits of the Two Species Lotka-Volterra System

Fig. 1 depicts the state orbits of the two species Lotka-Volterra system (7) . Thus, it is clear that the equilibrium $x^* = \begin{bmatrix} c/d \\ a/b \end{bmatrix}$ is stable.

Next, we suppose that the prey population is given as the output function of the system (7), *i.e.*

$$y = x_1 - x_1^* \tag{12}$$

The linearization of the output function (12) at the equilibrium $x = x^*$ is

$$C = \frac{\partial h}{\partial x}\left(x^*\right) = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{13}$$

From (9) and (13), the observability matrix for the Lotka-Volterra system (7) with output (12) is obtained as

$$\boldsymbol{O}(C, A) = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -bc/d \end{bmatrix}$$

which has full rank. Thus, by Theorem 1, we obtain the following result for the Lotka-Volterra system (7).

Theorem 3. The two species Lotka-Volterra system (7) with the output function (12) is locally observable at the equilibrium $x = x^*$. ■

Also, we have shown that the equilibrium $x = x^*$ of the Lotka-Volterra system (7) is Lyapunov stable. Thus, by Sundarapandian's Theorem (Theorem 2), we obtain the following result.

Theorem 4. The two species Lotka-Volterra system (7) with output function (12) has a local exponential observer of the form

$$\dot{z} = f(z) + K\left[y - h(z)\right], \tag{14}$$

where K is a gain matrix such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be found so that the error matrix $E = A - KC$ has arbitrarily assigned eigenvalues with negative real parts. ■

Example 1. Consider a predator-prey model given by

$$\begin{aligned} \dot{x}_1 &= x_1(6 - 1.5x_2) \\ \dot{x}_2 &= x_2(-3 + 0.5x_1) \end{aligned} \tag{15}$$

which has the positive equilibrium $x^* = \begin{bmatrix} 6, & 4 \end{bmatrix}^T$.

Suppose that the prey population is available for measurement. Thus, we can take as the output function

$$y = x_1 - x_1^* = x_1 - 6. \tag{16}$$

The linearization pair of the system (15)-(16) about the equilibrium $x = x^*$ is:

$$A = \begin{bmatrix} 0 & -9 \\ 2 & 0 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}.$$

We have already shown that the system linearization pair (C, A) is observable. Thus, the eigenvalues of $A - KC$ can be arbitrarily placed. Using Ackermann's formula [28], we can choose K so that $A - KC$ has the eigenvalues $\{-2, -2\}$. A simple calculation using MATLAB yields $K = \begin{bmatrix} 2 \\ -7 \end{bmatrix}$.

By Theorem 4, a local exponential observer for the given system (15)-(16) around $x = x^*$ is given by

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_1(6 - 1.5z_2) \\ z_2(-3 + 0.5z_1) \end{bmatrix} + \begin{bmatrix} 4 \\ 14/9 \end{bmatrix}\left[y - z_1 + 6\right] \tag{17}$$

Fig. 2 depicts the convergence of the observer states z_1 and z_2 of the system (17) to the plant states x_1 and x_2 of the Lotka-Volterra system (15). For simulation, we have taken the initial conditions as

$$x(0) = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \quad \text{and} \quad z(0) = \begin{bmatrix} 8 \\ 5 \end{bmatrix}.$$

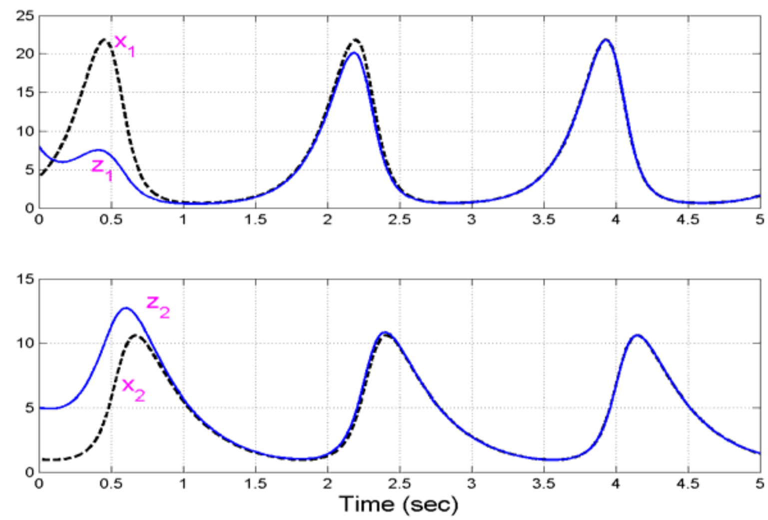

Figure 2. Local Exponential Observer for the Two Species Lotka-Volterra System (15)-(16)

IV. Exponential Observers for Three Species Lotka-Volterra Systems

In this section, we solve the problem of nonlinear observer design for three-species Lotka-Volterra systems, where the lowest level prey x_1 is preyed upon by a mid-level species x_2 which is, in turn, preyed upon by a top-level species x_3. Typical examples of such three-species Lotka-Volterra ecosystems are mouse-snake-owl, vegetation-hare-lynx and worn-robin-falcon ecosystems.

The three species Lotka-Volterra system is given by

$$\begin{aligned} \dot{x}_1 &= a\, x_1 - b\, x_1\, x_2 \\ \dot{x}_2 &= -c\, x_2 + d\, x_1 x_2 - e\, x_2 x_3 \\ \dot{x}_3 &= -f\, x_3 + g\, x_2 x_3 \end{aligned} \tag{18}$$

where $a, b, c, d, e, f, g > 0.$

The Lotka-Volterra system (18) has two equilibria, viz.

$$\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad x^* = \begin{bmatrix} c/d \\ a/b \\ 0 \end{bmatrix}.$$

The equilibrium at the origin is unstable and the equilibrium $x = x^*$ is stable under the assumption that $ga < fb.$ In this section, we shall study the nonlinear observer design problem around $x = x^*$.

We assume that the lowest level prey population x_1 is available for measurement. Thus, we take the output function of the system as

$$y = x_1 - x_1^*. \tag{19}$$

The linearization matrix of the plant dynamics (18) at $x = x^*$ is given by

$$A = \frac{\partial f}{\partial x}\left(x^*\right) = \begin{bmatrix} 0 & -bc/d & 0 \\ ad/b & 0 & -ae/b \\ 0 & 0 & -f + (ga/b) \end{bmatrix}$$

The linearization of the output function (19) at $x = x^*$ is given by

$$C = \frac{\partial h}{\partial x}\left(x^*\right) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

The observability matrix of the given system at $x = x^*$ is obtained as

$$\boldsymbol{O}(C, A) = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -bc/d & 0 \\ -ac & 0 & ace/d \end{bmatrix}$$

which has the determinant $-\dfrac{abc^2e}{d} \neq 0.$ Thus, the observability matrix $\boldsymbol{O}(C, A)$ has full rank.

Hence, by Theorem 1, we have proved the following result.

Theorem 5. The three-species Lotka-Volterra system (18) with the output function (19) is locally observable at $x = x^*$. ■

If we suppose that $-f + \dfrac{ga}{b} < 0$ or equivalently that $ga < fb,$ then by Lyapunov stability theory, it can be easily established that the equilibrium $x = x^*$ is Lyapunov stable. Thus, we can apply Sundarapandian's theorem (Theorem 2) to the three-species Lotka-Volterra system (18)-(19) and arrive at the following result.

Theorem 6. Let $ga < fb.$ Then the Lotka-Volterra system (18)-(19) has a local exponential observer of the form

$$\dot{z} = f(z) + K\left[y - h(z)\right], \tag{20}$$

where K is a gain matrix such that $A - KC$ is Hurwitz. Since (C, A) is observable, a gain matrix K can be found so that the error matrix $E = A - KC$ has arbitrarily assigned eigenvalues with negative real parts. ■

Example 2. Consider a three-species Lotka-Volterra population model given by

$$\begin{aligned} \dot{x}_1 &= 2x_1 - 0.5x_1x_2 \\ \dot{x}_2 &= -2x_2 + x_1x_2 - 0.6x_2x_3 \\ \dot{x}_3 &= -3x_3 + 0.4x_2x_3 \end{aligned} \tag{21}$$

which has the non-trivial equilibrium $x^* = \begin{bmatrix} 2, & 4, & 0 \end{bmatrix}^T$.

Here, the condition $ga < fb$ is satisfied because $a = 2, b = 0.5, f = 3$ and $g = 0.4.$

Thus, the equilibrium $x = x^*$ of the system (21) is Lyapunov stable as depicted in Fig. 3.

We assume that the lowest level prey population x_1 is available for measurement. Thus, we take the output function of the system as

$$y = x_1 - x_1^* = x_1 - 2. \tag{22}$$

T he linearization pair of the three-species Lotka-Volterra system (21)-(22) at x^* is obtained as

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 4 & 0 & -2.4 \\ 0 & 0 & -1.4 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

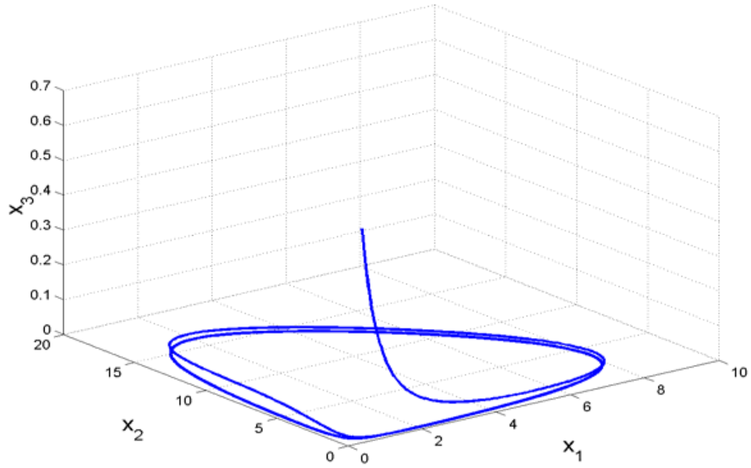

Figure 3. State Orbits of the Three-Species Lotka-Volterra System (21)

As already shown earlier, the system pair (C, A) is observable. Thus, we can find an observer gain matrix K such that the eigenvalues of the error matrix $E = A - KC$ can be arbitrarily assigned in the stable region of the complex plane. In particular, using Ackermann's formula, we can find a gain matrix K so that $E = A - KC$ has the eigenvalues $\{-2, -2, -2\}$. A simple calculation yields $K = \begin{bmatrix} 4.6, & -1.56, & 0.09 \end{bmatrix}^T$.

Thus, by Theorem 6, a local exponential observer for the Lotka-Volterra system (21)-(22) near x^* is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} 2\, z_1 - 0.5\, z_1\, z_2 \\ -2\, z_2 + z_1 z_2 - 0.6\, z_2 z_3 \\ -3\, z_3 + 0.4\, z_2 z_3 \end{bmatrix} + \begin{bmatrix} 4.6 \\ -1.56 \\ 0.09 \end{bmatrix} \begin{bmatrix} y - z_1 + 2 \end{bmatrix} \tag{23}$$

Fig. 4 depicts the convergence of the observer states z_1, z_2 and z_3 of the observer system (23) to the plant states x_1, x_2 and x_3 of the three-species Lotka-Volterra system (21).

For simulation, we have taken the initial conditions as $x(0) = \begin{bmatrix} 4, & 6, & 2 \end{bmatrix}^T$ and $z(0) = \begin{bmatrix} 8, & 2, & 5 \end{bmatrix}^T$.

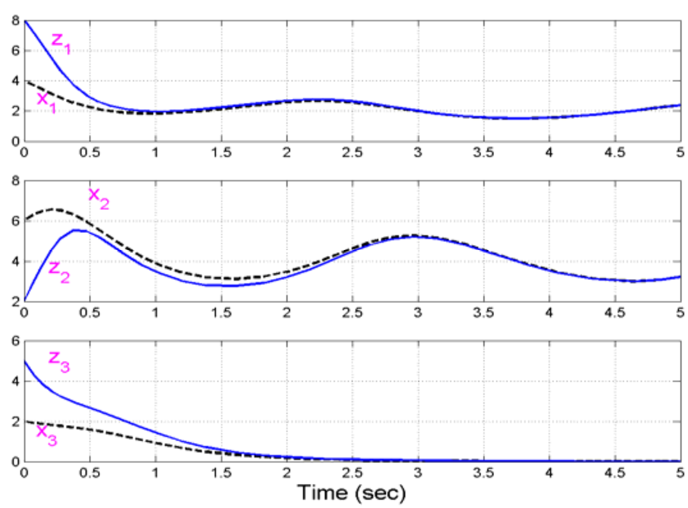

Figure 4. Local Exponential Observer for the Three Species Lotka-Volterra System (21)-(22)

V. EXPONENTIAL OBSERVERS FOR FOUR SPECIES LOTKA-VOLTERRA SYSTEMS

In this section, we solve the problem of nonlinear observer design for four-species Lotka-Volterra systems, where the lowest level prey x_1 is preyed upon by a mid-level species x_2 which is, in turn, preyed upon by a mid-level species x_3 and which is, in turn, preyed upon by a top-level species x_4. Typical examples of such four-species Lotka-Volterra ecosystems are vegetation-mouse-snake-owl, worm-frog-snake-falcon, etc.

The four species Lotka-Volterra system is given by

$$\begin{aligned} \dot{x}_1 &= a\, x_1 - b\, x_1\, x_2 \\ \dot{x}_2 &= -c\, x_2 + d\, x_1 x_2 - e\, x_2 x_3 \\ \dot{x}_3 &= -f\, x_3 + g\, x_2 x_3 - h\, x_3 x_4 \\ \dot{x}_4 &= -\alpha\, x_4 + \beta\, x_3 x_4 \end{aligned} \tag{24}$$

where $a, b, c, d, e, f, g, h, \alpha, \beta > 0.$

The system (24) has two equilibria, *viz.* $\mathbf{0} = \begin{bmatrix} 0, & 0, & 0, & 0 \end{bmatrix}^T$ and $x^* = \begin{bmatrix} c/d, & a/b, & 0, & 0 \end{bmatrix}^T$.

The equilibrium at the origin is unstable and the equilibrium $x = x^*$ is stable under the assumption that $ga < fb.$ In this section, we shall study the nonlinear observer design problem around $x = x^*$.

We assume that the lowest level prey population x_1 is available for measurement. Thus, we take the output function of the system as

$$y = x_1 - x_1^*. \tag{25}$$

The linearization matrix of the plant dynamics (24) at $x = x^*$ is given by

$$A = \frac{\partial f}{\partial x}\left(x^*\right) = \begin{bmatrix} 0 & -bc/d & 0 & 0 \\ ad/b & 0 & -ae/b & 0 \\ 0 & 0 & -f + (ag/b) & 0 \\ 0 & 0 & 0 & -\alpha \end{bmatrix}$$

The linearization of the output function (25) at $x = x^*$ is given by

$$C = \frac{\partial h}{\partial x}\left(x^*\right) = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

It is easy to show that the observability matrix $\mathbf{O}(C, A)$ has rank three, *i.e.* the system (24)-(25) is not observable. However, since the unobservable mode $\lambda = -\alpha$ is stable, it is easy to show that the system (24)-(25) is detectable, *i.e.* there exists a matrix K such that $A - KC$ is Hurwitz. Hence, we can apply Sundarapandian's theorem (Theorem 2) to the four-species Lotka-Volterra system (24)-(25) and arrive at the following result.

Theorem 7. Let $ga < fb.$ Then the Lotka-Volterra system (24)-(25) has a local exponential observer of the form

$$\dot{z} = f(z) + K\left[y - h(z)\right], \tag{26}$$

where K is a gain matrix such that $A - KC$ is Hurwitz. ■

Example 3. Consider a four-species Lotka-Volterra population model given by

$$\begin{aligned} \dot{x}_1 &= 3x_1 - 0.5x_1x_2 \\ \dot{x}_2 &= -4x_2 + 2x_1x_2 - 0.5x_2x_3 \\ \dot{x}_3 &= -4x_3 + 0.2x_2x_3 - 0.5x_3x_4 \\ \dot{x}_4 &= -2x_4 + x_3x_4 \end{aligned} \tag{27}$$

which has the non-trivial equilibrium $x^* = \begin{bmatrix} 2, & 6, & 0, & 0 \end{bmatrix}^T$.

Here, the condition $ga < fb$ is satisfied because $a = 3,\ b = 0.5,\ f = 4$ and $g = 0.2.$

Thus, the equilibrium $x = x^*$ of the system (27) is Lyapunov stable as depicted in Fig. 5.

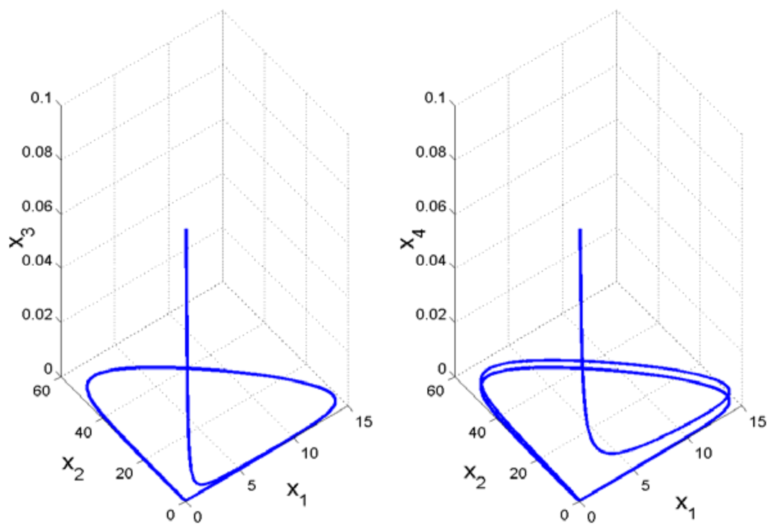

Fig. 5. State Orbits of the Four-Species Lotka-Volterra System (27)

We assume that the lowest level prey population x_1 is available for measurement. Thus, we take the output function of the system as

$$y = x_1 - x_1^* = x_1 - 2. \tag{28}$$

T he linearization pair of the four-species Lotka-Volterra system (27)-(28) at x^* is obtained as

$$A = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 12 & 0 & -3 & 0 \\ 0 & 0 & -2.8 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}.$$

If we take $K = \begin{bmatrix} 3.2, & 8.96, & -0.1707, & -2 \end{bmatrix}^T$, then it is easy to check that $E = A - KC$ is Hurwitz with the eigenvalues $\{-2, -2, -2, -2\}$.

Thus, by Theorem 7, a local exponential observer for the Lotka-Volterra system (27)-(28) near x^* is

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \\ \dot{z}_3 \\ \dot{z}_4 \end{bmatrix} = \begin{bmatrix} 3z_1 - 0.5z_1z_2 \\ -4z_2 + 2z_1z_2 - 0.5z_2z_3 \\ -4z_3 + 0.2z_2z_3 - 0.5z_3z_4 \\ -2z_4 + z_3z_4 \end{bmatrix} + \begin{bmatrix} 3.2 \\ 8.96 \\ -0.1707 \\ -2 \end{bmatrix} \left[y - z_1 + 2 \right]. \tag{29}$$

Fig. 6 depicts the convergence of the observer states z_1, z_2, z_3 and z_4 of the observer system (29) to the plant states x_1, x_2 and x_3 of the four-species Lotka-Volterra system (27).

For simulation, we have taken the initial conditions as

$$x(0) = \begin{bmatrix} 3 \\ 8 \\ 5 \\ 4 \end{bmatrix} \quad \text{and} \quad z(0) = \begin{bmatrix} 1 \\ 4 \\ 7 \\ 6 \end{bmatrix}.$$

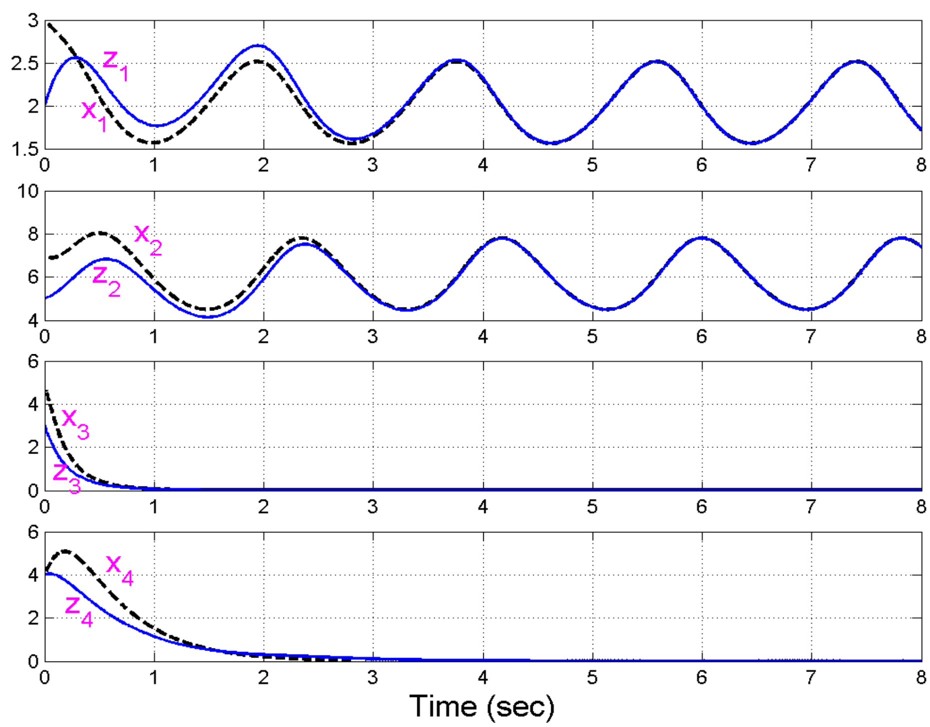

Figure 6. Local Exponential Observer for the Four-Species Lotka-Volterra System (27)-(28)

VI. Conclusions

For many real-world problems of population and conservation ecology, an efficient monitoring system is of great importance. In this paper, the methodology based on Sundarapandian's theorem (2002) has been suggested for the monitoring of multi-species Lotka-Volterra systems, viz. two-species, three-species and four-species systems. Theorems have been derived in detail for each of the multi-species Lotka-Volterra systems studied in this paper and numerical examples have been provided to illustrate the effectiveness of the proposed exponential observer design for the multi-species Lotka-Volterra population biology systems.

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