NEW NUMERICAL ALGORITHMS FOR MINIMIZATION OF NONLINEAR FUNCTIONS

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Abstract

In this paper, we propose few new algorithms of third order convergence for minimization of nonlinear functions which is based on geometric construction of iteration functions of order three to develop cubically convergent iterative methods. Then comparative study among the new algorithms and Newton's algorithm is established by means of examples.

Key words: Nonlinear functions; Minimization; Newton's method; Third order of convergence

1. INTRODUCTION

Optimization problems with or without constraints arise in various fields such as science, engineering, economics especially in management sciences, engineering design, operation research, computer science, financial management, etc., where numerical information is processed. In recent years, many problems in business situations and engineering designs have been modeled as an optimization problem for taking optimal decisions. In fact, numerical optimization techniques have made deep in to almost all branches of engineering and mathematics.

Several methods [8, 13, 17]are available for solving unconstrained minimization problems. All the unconstrained minimization methods are iterative in nature and hence they start from an initial trial solution and proceed towards the minimum point in a sequential manner. To solve unconstrained nonlinear minimization problems arising in the diversified field of engineering and technology, we have several methods to get solutions. For instance, multi- step nonlinear conjugate gradient methods [4], a scaled nonlinear conjugate gradient algorithm[1], a method called, ABS-MPVT algorithm [14] are used for solving unconstrained optimization problems. Newton's method [15] is used for various classes of optimization problems, such as unconstrained minimization problems, equality constrained minimization problems. A proximal bundle method with inexact data [18] is used for minimizing unconstrained non smooth convex function. Implicit and adaptive inverse preconditioned gradient method [3] is used for solving nonlinear minimization problems. A new algorithm [7] is used for solving unconstrained optimization problem with the form of sum of squares minimization. A derivative based algorithm [10] is used for a particular class of mixed variable optimization problems. A globally derivative – free decent method [16] is used for nonlinear complementarity's problems.

Vinay Kanwar ***et al.*** [19] introduced new algorithms called, external touch technique and orthogonal intersection technique for solving the nonlinear equations. Mamta et. al. [11] proposed a class of quadratic ally convergent iteration formulae. Weerakoon.S, Fernando.G.I.[20] introduced a variant of Newton's method with accelerated third order convergence. X-Y.Wu.[21] proposed a new continuation Newton like method with third-order convergence. Kou. et al [9] introduced a modification of Newton's method with third order convergence. Homeier[5, 6] proposed a modified Newton's method for root finding with cubic convergence and Newton-type methods with cubic convergence. Miquel Grau-Sanchez[12] proposed improvements of the efficiency of some three step iterative like Newton's methods. Recently, Changbum Chun[2] introduced a geometric construction of iterative functions of order three to develop cubically convergent iterative methods for solving nonlinear equations. This construction can be applied to any iteration function of order two to develop an iteration function of order three.

In this paper, we introduce few new algorithms of third order convergence for minimization of nonlinear functions and comparative study is established among the new algorithms with Newton's algorithm by means of examples.

2. NEW ALGORITHMS

In this section, we will introduce six new algorithms for minimization nonlinear real valued and twice differentiable real functions using the concept of geometric construction of iteration functions of order three to develop cubically convergent iterative methods.

Consider the nonlinear optimization problem: Minimize $\{f(x),\ x \in R,\ f : R \to R\}$ where f is a nonlinear twice differentiable function.

2.1. *New methods*

Consider the function $G(x) = x - (g(x)/g'(x))$ where $g(x) = f'(x)$. Here $f(x)$ is the function to be minimized. $G'(x)$ is defined around the critical point x^* of $f(x)$ if $g'(x^*) = f''(x^*) \neq 0$ and is given by $G'(x) = g(x)g''(x)/g'(x)$.

If we assume that $g''(x^*) \neq 0$, we have $G'(x^*) = 0$ iff $g(x^*) = 0$.

We consider iterative methods to solve non linear equation g(x) = 0 that uses g and g' but not the higher derivatives of g. Here we develop iterative methods with at least cubic convergence that does not require the computation of second derivatives. This approach is based on a simple geometric construction and can use any second order iterative method in deriving a third order method and construct iteration functions of order three from the given iteration functions of order two based on a geometric observation. In the sequel, whenever we mention that an iteration function ϕ is of order p, the corresponding iterative method is defined by

$x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, \ldots$ is of convergence order p. For notational convergence, we indicate that ϕ is an iteration function whose order is p by writing $\phi \in I_p$

The proposed scheme is constructed geometrically as follows. Let x be a guess for α and let ϕ be an iteration function with $\phi \in I_2$. We consider the function $\bar{\phi}$ satisfying $\dfrac{\phi(x) + \bar{\phi}(x)}{2} = x$ ie., $\bar{\phi}(x) = 2x - \phi(x)$. Notice that x is the point which bisects that segment which lies between $[\phi(x), 0]$ and $[\bar{\phi}(x), 0]$. As the next guess for α, we consider the intersection $\psi(x)$ with the x-axis of the line through $(\bar{\phi}(x),\ g[\bar{\phi}(x)])$ which is parallel to the tangent line at (x, g(x)). This $\psi(x)$ can be written as follows.

$\psi(x) = \bar{\phi}(x) - \dfrac{g[\bar{\phi}(x)]}{g'(x)}$. The following theorem shows that the iteration function ψ constructed in this way is of order three.

2.1.1. *Theorem*: Let $\alpha \in J$ be a simple zero of sufficiently differentiable function g : J → R for an open interval J. ie., $g(\alpha) = 0$ and $g'(\alpha) \neq 0$, ϕ be an iteration function with $\phi \in I_2$, such that $\phi^{(3)}$ is continuous in a neighborhood of α . Let $\bar{\phi}$ be the function satisfying $\dfrac{\phi(x) + \bar{\phi}(x)}{2} = x$ and let

$\psi(x) = \bar{\phi}(x) - \dfrac{g[\bar{\phi}(x)]}{g'(x)}$ __(1) Then $\psi \in I_3$.

Proof: We will prove that $\psi(\alpha) = \alpha,\ \psi'(\alpha) = \psi''(\alpha) = 0$ .Observe that $\bar{\phi}(\alpha) = \alpha,\ \bar{\phi}'(\alpha) = 2,\ \bar{\phi}^{(j)}(\alpha) = -\phi^{(j)}(\alpha),\ j \geq 2$. It is easy to very that $\psi(\alpha) = \alpha,\ \psi'(\alpha) = 0$.

Rewrite the definition of $\psi(x)$ as $g'(x)\psi(x) = \bar{\phi}(x)g'(x) - g[\bar{\phi}(x)]$. ___(2)

Let k be any integer. Differentiating the above equation (2) k times with respect to x yields

$$\sum_{j=0}^{k} C(k,\ j)\, g^{(k-j+1)}(x)\psi^{(j)}(x) = \sum_{j=0}^{k} C(k,\ j)\, g^{(k-j+1)}(x)\, \bar{\phi}^{(j)}(x) - \frac{d^k g[\bar{\phi}(x)]}{dx^k} \quad \_\_\_(3)$$

where $C(k,\ j)$ is a binomial coefficient. Set $x = \alpha$ and $k = 2$ in the above equation (3) we have $g'(\alpha)\,\psi''(\alpha) = 4\, g''(\alpha) + g'(\alpha)\,\bar{\phi}''(\alpha) - \left.\frac{d^2 g[\bar{\phi}(x)]}{dx^k}\right|_{x=\alpha}$ and

$\left.\frac{d^2 g[\bar{\phi}(x)]}{dx^k}\right|_{x=\alpha} = 4\, g''(\alpha) + g'(\alpha)\,\bar{\phi}''(\alpha)$. Hence $g'(\alpha)\,\psi''(\alpha) = 0$.

Since $g'(\alpha) \neq 0$, we conclude that $\psi''(\alpha) = 0$. □

An elementary computation for $\psi^{(3)}(\alpha)$ yields $\psi^{(3)}(\alpha) = \frac{3g''(\alpha)\phi''(\alpha) - 2g^{(3)}(\alpha)}{g'(\alpha)}$ and in general $\psi^{(3)}(\alpha) \neq 0$. So in most cases the iteration function defined by (1) is of order three.

The above theorem is independent of the structure of the iteration function of order two involved, so any iteration function of order two gives rise to an iteration function of order three. In this article, several iteration functions of order two are considered to illustrate how the above theorem can be employed as means of deriving cubically convergent methods. Some of the obtained methods are then compared with some of the existing third-order methods.

2.1.2. *New method – (1)*

Consider Newton's iteration function defined by $\phi(x) = x - \frac{g(x)}{g'(x)}$. In this case $\bar{\phi}(x) = x + \frac{g(x)}{g'(x)}$. By the above theorem, the iteration function ψ defined by

$\psi(x) = x - \frac{g(x + g(x)/g'(x)) - g(x)}{g'(x)}$ is of order three. Therefore we obtain the iterative method with third order convergence [9] is given by

$$x_{n+1} = x_n - \frac{g(x_n + g(x_n)/g'(x_n)) - g(x_n)}{g'(x_n)} \qquad \text{(I)}$$

Since $g(x) = f'(x)$, the equation (I) becomes

New Algorithm – (1)

$$x_{n+1} = x_n - \frac{f'(x_n + f'(x_n)/f''(x_n)) - f'(x_n)}{f''(x_n)} \qquad \text{(II)}$$

2.1.3. *New method – (2)*

Let $\phi(x) = x - g(x)/(g(x) + g'(x))$ This iteration is of order two [21]. By the above theorem with $\bar{\phi}(x) = x + g(x)/(g(x) + g'(x))$ implies that the iteration function ψ defined by

$\psi(x) = x + \frac{g(x)}{g(x) + g'(x)} - \frac{g(x + g(x)/(g(x) + g'(x))}{g'(x)}$ is of order three. Hence we obtain the new iterative method with third order convergence is given by

$$x_{n+1} = z_n - \frac{g(z_n)}{g'(x_n)} \quad \text{where } z_n = x_n + \frac{g(x_n)}{g(x_n) + g'(x_n)} \qquad \text{(III)}$$

Since $g(x) = f'(x)$, the equation (III) becomes

New Algorithm – (2)

$$x_{n+1} = z_n - \frac{f'(z_n)}{f''(x_n)} \quad \text{where} \quad z_n = x_n + \frac{f'(x_n)}{f'(x_n) + f''(x_n)} \qquad \text{(IV)}$$

2.1.4. ***New method – (3)***

Let $\phi(x) = x - g(x)\,g'(x)/(g^2(x) + g'^2(x))$. This iteration is of order two[11].. By the above theorem with $\overline{\phi}(x) = x + g(x)\,g'(x)/(g^2(x) + g'^2(x))$ implies that the iteration function ψ defined by

$$\psi(x) = x + \frac{g(x)\,g'(x)}{g^2(x) + g'^2(x)} - \frac{g(x + g(x)\,g'(x)/(g^2(x) + g'^2(x)))}{g'(x)}$$ is of order three. Hence we obtain the new iterative method with third order convergence is given by

$$x_{n+1} = z_n - \frac{g(z_n)}{g'(x_n)} \quad \text{where} \quad z_n = x_n + \frac{g(x_n)\,g'(x_n)}{g^2(x_n) + g'^2(x_n)} \qquad \text{(V)}$$

Since $g(x) = f'(x)$, the equation (V) becomes

New Algorithm – (3)

$$x_{n+1} = z_n - \frac{f'(z_n)}{f''(x_n)} \quad \text{where} \quad z_n = x_n + \frac{f'(x_n)\,f''(x_n)}{f'^2(x_n) + f''^2(x_n)}$$

2.1.5. New method – (4)

Based on the method of Weerakoon and Fernando[20], we have the following iterative method which is the third order of convergence

$$x_{n+1} = x_n - \frac{2\,g(x_n)}{g'(x_n) + g'(x_n - g(x_n)/g'(x_n))} \qquad \text{(VI)}$$

Since $g(x) = f'(x)$, the equation (VI) becomes

New Algorithm – (4)

$$x_{n+1} = x_n - \frac{2\,f'(x_n)}{f''(x_n) + f''(x_n - f'(x_n)/f''(x_n))} \qquad \text{(VII)}$$

2.2. ***New method – (5)***

Based on the mid-point rule, we have the following iterative method [2] which is the third order of convergence

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n - g(x_n)/(2\,g'(x_n)))} \qquad \text{(VIII)}$$

Since $g(x) = f'(x)$, the equation (VIII) becomes

New Algorithm-(5)

$$x_{n+1} = x_n - \frac{f'(x_n)}{f''(x_n - f'(x_n)/(2\,f''(x_n)))}$$

2.3. New method – (6)

Based on the method of Homeier[5], we have the following iterative method which is also the third order of convergence

$$x_{n+1} = x_n - \frac{g(x_n)}{2}\left(\frac{1}{g'(x_n)} + \frac{1}{g'(x_n - g(x_n)/g'(x_n))}\right) \qquad \text{(IX)}$$

Since $g(x) = f'(x)$, the equation (IX) becomes

New Algorithm – (6)

$$x_{n+1} = x_n - \frac{f'(x_n)}{2}\left(\frac{1}{f''(x_n)} + \frac{1}{f''(x_n - f'(x_n)/f''(x_n))}\right) \qquad \text{(X)}$$

3. Numerical Results

Example.3.1: Consider the function $f(x) = x^3 - 2x - 5$. The minimized value of the function is 0.816497. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Table – I : shows a comparison between the New iterative methods and Newton's method

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	3	5	5
2.	New Algorithm – (1)	2	3	4
3	New Algorithm – (2)	2	4	4
4	New Algorithm – (3)	2	4	4
5	New Algorithm – (4)	2	3	3
6	New Algorithm – (5)	2	3	3
7	New Algorithm – (6)	2	3	3

Example.3.2: Consider the function $f(x) = x^4 - x - 10$. The minimized value of the function is 0.629961. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Table – II: shows a comparison between the New iterative methods and Newton's method

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	4	6	7
2	New Algorithm – (1)	3	4	5
3	New Algorithm – (2)	4	5	6
4	New Algorithm – (3)	3	5	5
5	New Algorithm – (4)	3	4	5
6	New Algorithm – (5)	3	4	5
7	New Algorithm – (6)	3	4	4

Example 3.3: Consider the function $f(x) = xe^x - 1$. The minimized value of the function is -1. The following table depicts the number of iterations needed to converge to the minimized value for all the new algorithms with three initial values $x_0 = 1$, $x_0 = 2$, and $x_0 = 3$.

Table – III: shows a comparison between the New iterative methods and Newton's method

Sl. No	Methods	For initial value x_0 =1.000000	For initial value x_0 =2.000000	For initial value x_0 =3.000000
1	Newton's Algorithm	7	8	10
2	New Algorithm – (1)	5	5	6
3	New Algorithm – (2)	5	7	8
4	New Algorithm – (3)	5	6	7
5	New Algorithm – (4)	5	6	7
6	New Algorithm – (5)	5	5	6
7	New Algorithm – (6)	4	5	5

4. CONCLUSION

In this paper, we have introduced six new numerical algorithms for minimizing nonlinear unconstrained optimization problems and compared with Newton's method. It is clear from numerical results that the rate of convergence of new algorithms is better than Newton's algorithm. In real life problems, the variables can not be chosen arbitrarily rather they have to satisfy certain specified conditions called constraints. Such problems are known as constrained optimization problems. In near future, we have a plan to extend the proposed new algorithms to constrained optimization problems.

REFERENCES

[1] N.Andrei , "***A scaled nonlinear conjugate gradient algorithm for unconstrained minimization",*** Optimization 57(4), (2008), 549 – 570.

[2] Changbum Chun, "A geometric construction of iterative functions of order three to solve non linear equations", Computers and Mathematics with Applications, 53 (2007), 972 – 976.

[3] J.P.Chehab and M.Raydon, ***"Implicit and adaptive inverse preconditioned gradient methods for nonlinear problems "***, Applied Numerical Mathematics, 55(1), (2005), 32 -47.

[4] J.A.Ford, Y.Narushima,. and H.Yabe," ***Multi-step nonlinear conjugate gradient methods for constrained minimization"***, Computational Optimization and application 40(2), (2008), 191 -216.

[5] H.H.H.Homeier, ***"A modified Newton method for root finding with cubic Convergence",*** J. Comput. Appl. Math., 157(2003), 227 – 230.

[6] H.H.H.Homeier, ***"On Newton-type methods with cubic convergence"*** J. Comput. Appl. Math., 176 (2005), 425-432.

[7] Y.Hu, H.Su, and J. Chu, "***A new algorithm for unconstrained Optimization Problem with the form of sum of squares minimization"***, Conference proceedings - IEEE international conference on systems, man and cybernetics 7 (2004), 6108 – 6112.

[8] P.Kanniappan and K.Thangavel, ***" Unidimensional Search Scheme Using Identric Mean for Optimization Problems ",*** Opsearch, 38 (2), (2001) 197-209.

[9] J. Kou., Y. Li., X.Wang, ***"A modification of Newton's method with third order convergence",*** Appl. Math. Comput., 181(2006), 1106 – 1111.

[10] S. Lucid and V. Piccialli ***" A derivative based algorithm for a particular class of mixed variable optimization problems "***, Optimization Methods and Software 19(3-4), (2004), 371-387.

[11] Mamta, V. Kanwar, V.K. Kukreja, S. Singh, " ***On a class of quadratically*** convergent iteration formulae", Appl. Math. Comput., 166(2005), 633 – 637.

[12] Mique Grau-Sanchez, ***"Improvements of the efficiency of some three step iterative like Newton's method"***, Numer. Math. 107(2007), 131 – 146.

[13] Mohan C Joshi and Kannan M Moudgalya,***" Optimization theory and Practice "*** Narosa Publication House, New Delhi, 2004.

[14] L.P.Pang, E.Spedicato, Z.Q.Xia, and W.Wang, ***"A method for solving the system of linear equations and linear inequalities "***, Mathematical and Computer Modelling 46(5-6) (2007), 823 – 836.

[15] B.T.Polyak, ***"Newton's method and its use in optimization "***, European Journal of Operational research 181(3) (2007), 1086 – 1096.

[16] H.D.Qi, "***A globally derivative-free descent method for nonlinear complementarity problems",*** Journal of Computational Mathematics, 18(3) (2000), 251-264.

[17] G.V.Reklaitis, A.Ravindran, and K.M.Ragsdell, ***"Engineering optimization methods and applications "*** ,John Wiley and sons, New York, 1983..
[18] J.Shen, Z.Q. Xia, and L.P. Pang, ***"A proximal bundle method with inexact data for convex non differentiable minimization"***, Nonlinear Analysis, Theory, Methods and Applications 66(9) (2007), 2016 -2027.
[19] Vinay Kanwar, Sukhjit Singh, J.R.Sharma, and Mamta, ***"New numerical techniques for solving non-linear equations"***, Indian Journal of Pure and Applied Mathematics 34(9) (2003), 1339 – 1349.
[20] S.Weerakoon, G.I. Fernando, ***"A variant of Newton's method with accelerated third-order convergence*** ", Appl. Math. Lett., 17(2000), 87 – 93.
[21] X.Y.Wu, ***" A new continuation Newton-like method and its deformation "***, Appl. Math. Comput., 112 (2000), 75 – 78..