Co-Recursively Enumerable Triods with Computable Endpoints

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***Abstract*—Recursive sets in the Euclidean space are those sets which can be effectively approximated by finitely many points for an arbitrary given precision. On the other hand, co-recursively enumerable sets are those sets whose complements can be effectively covered by open balls. If a set is recursive, then it is co-recursively enumerable, however the converse is not true in general. In this paper we investigate the subsets of the Euclidean space called triods and we prove that each co-r.e. triod with computable endpoints is recursive.**

Keywords-recursive set, co-recursively enumerable set, triod

I. Introduction

A real number x is said to be **computable** ([7, 8]) if there exist recursive functions $a,b,c:\mathbf{N}\to\mathbf{N}$, $b(k)\neq 0$, $\forall k\in\mathbf{N}$, such that

$$\left|x-(-1)^{c(k)}\frac{a(k)}{b(k)}\right|<2^{-k},$$

$\forall k\in\mathbf{N}$. In other words, x is computable if there exists an algorithm which, for a given $k\in\mathbf{N}$, gives a rational number which is a $2^{-k}-$approximation of x.

A sequence of real numbers (x_i) is said to be **computable** if there exist recursive functions $a,b,c:\mathbf{N}^2\to\mathbf{N}$, $b(i,k)\neq 0$, $\forall i,k\in\mathbf{N}$, such that

$$\left|x_i-(-1)^{c(i,k)}\frac{a(i,k)}{b(i,k)}\right|<2^{-k},$$

$\forall i,k\in\mathbf{N}$.

Let $n\geq 1$. A point $x\in\mathbf{R}^n$, $x=(x_1,\dots,x_n)$ is said to be **computable** if $x_1,\dots,x_n$ are computable numbers. We similarly define the notion of a computable sequence in $\mathbf{R}^n$.

The central notion of this paper is the notion of a recursive (computable) subset of the Euclidean space $\mathbf{R}^n$. A closed subset S of $\mathbf{R}^n$ is said to be **recursive** if $S=\emptyset$ or if the sequence of real numbers $(d(x_i,S))$ is computable for each computable sequence (x_i) in $\mathbf{R}^n$. Here $d(x_i,S)$ denotes the distance from the point x_i to the set S, i.e. the number $\inf\{d(x_i,s)\mid s\in S\}$ (d is the Euclidean metric on $\mathbf{R}^n$).

If a nonempty set $S\subseteq\mathbf{R}^n$ is recursive, then S contains a recursive point, moreover recursive points of S are dense in S. In fact, it can be shown that the following result holds (for simplicity we assume here that S is bounded): a closed nonempty $S\subseteq\mathbf{R}^n$ is recursive if and only if there exists a computable sequence (x_i) in $\mathbf{R}^n$ such that $x_i\in S$, $\forall i\in\mathbf{N}$, and a recursive function $f:\mathbf{N}\to\mathbf{N}$ such that

$$S\subseteq\bigcup_{i=0}^{f(k)}B(x_i,2^{-k}), \tag{1}$$

$\forall k\in\mathbf{N}$; here, for $a\in\mathbf{R}^n$ and $r>0$, we denote by $B(a,r)$ the open ball of radius r centered at a, i.e. $B(a,r)=\{b\in\mathbf{R}^n\mid d(a,b)<r\}$. The relation (1) means that any point of S is $2^{-k}-$close to some of the points $x_0,\dots,x_{f(k)}$, hence the finite set of points $\{x_0,\dots,x_{f(k)}\}$ represents a $2^{-k}-$approximation of S.

Recursive sets can be considered as those sets which can be displayed by a physical computer for an arbitrary given resolution (as discussed in [1]).

There is also the notion of a co-recursively enumerable set, which is weaker than the notion of a recursive set. A closed set $S\subseteq\mathbf{R}^n$ is said to be **co-recursively enumerable** (co-r.e.) if the complement of S can be covered effectively by open balls, i.e. if there exist a computable sequence (x_i) in $\mathbf{R}^n$ and a computable sequence (r_i) of positive real numbers such that

$$\mathbf{R}^n \setminus S = \bigcup_{i\in\mathbf{N}} B(x_i, r_i).$$

It is easy to see that each recursive set is co-r.e. On the other hand, a co-r.e. set need not be recursive, moreover there exists a co-r.e. set which does not contain any recursive point.

Although the implication

$$S\ co-r.e. \Rightarrow S\ recursive \qquad (2)$$

fails to be true in general, it can be shown that under certain topological assumptions (2) holds. For example, if S is a topological circle or an arc with computable endpoints, then (2) holds ([5, 4]).

In this paper we prove that (2) also holds whenever S is a triod with computable endpoints. By a triod in $\mathbf{R}^n$ (see [3]) we mean a set homeomorphic to

$$T = ([-1,1]\times\{0\}) \cup (\{0\}\times[0,1]). \qquad (3)$$

Hence $S \subseteq \mathbf{R}^n$ is a triod if and only if there exists a continuous injection $f: T \to \mathbf{R}^n$ such that $f(T) = S$.

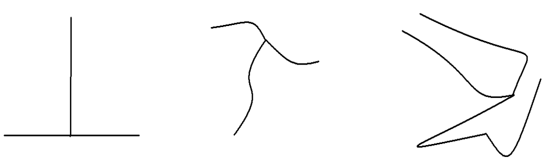

Figure 1. Examples of triods

If S is a triod, then a point $x \in S$ is called endpoint if $S\setminus\{x\}$ is connected. If $f: T \to S$ is a continuous injection, then $f(-1,0)$, $f(1,0)$ and $f(0,1)$ are all endpoints of S. Hence each triod has exactly three endpoints.

II. CO-R.E. TRIODS

In general, co-r.e. triods need not be recursive, as the following example shows.

Example 1 Let $\gamma \in \langle 0,1\rangle$ be a non-computable number such that the set $[\gamma,1]$ is co-r.e. in $\mathbf{R}$ (Example 2.2. in [5]). Then $[\gamma,1]\times\{0\}$ is co-r.e. in $\mathbf{R}^2$ (Example 4 in [4]) and therefore the set $S = ([\gamma,1]\times\{0\}) \cup (\{1\}\times[-1,1])$ is co-r.e. as the union of co-r.e. sets. Clearly, S is a triod, however S is not recursive since $d((0,0),S) = \gamma$, which is a non-computable number.

Let us suppose now that S is a co-r.e. triod in $\mathbf{R}^n$ with computable endpoints a, b and c. Hence there exists an effective procedure which lists open balls which cover the complement of S and, moreover, for each of the points a, b, c there exists an algorithm which computes that point for an arbitrary given precision. We want to prove that S is recursive. The idea is to find an effective procedure which, for a given $k \in \mathbf{N}$, gives a sequence of sets $A_0, \dots, A_m$ with the following properties:

(I) $\operatorname{diam} A_i < 2^{-k}$, $\forall i \in \{0,\dots,m\}$;

(II) $S \subseteq A_0 \cup \dots \cup A_m$;

(III) $A_i \cap S \neq \emptyset$, $\forall i \in \{0,\dots,m\}$.

Here $\operatorname{diam} A_i$ denotes the diameter of the set A_i, i.e. the number $\sup\{d(x,y) \mid x, y \in A_i\}$. Why are these three properties important? First, we have the following proposition, which is an easy consequence of the triangle inequality (see also Proposition 6 in [4]).

Proposition 1 *Let $\varepsilon > 0$ and let X and Y be nonempty subsets of $\mathbf{R}^n$. Suppose that for each $x \in X$ there exists $y \in Y$ such that $d(x,y) < \varepsilon$ and for each $y \in Y$ there exists $x \in X$ such that $d(y,x) < \varepsilon$. Then*

$$|d(z,X) - d(z,Y)| \le \varepsilon,$$

for each $z \in \mathbf{R}^n$.

Now, if we have sets $A_0, \dots, A_m$ with properties (I)-(III), then for each $x \in S$ there exists $y \in A_0 \cup \dots \cup A_m$ such that $d(x,y) < 2^{-k}$, which follows trivially from (II), and for each $y \in A_0 \cup \dots \cup A_m$ there exists $x \in S$ such that $d(y,x) < 2^{-k}$, which follows easily from (III) and (I). Therefore, by Proposition 1,

$$|d(z,S) - d(z, A_0 \cup \dots \cup A_m)| \le 2^{-k},$$

for each $z \in \mathbf{R}^n$. This means that if (x_i) is a computable sequence in $\mathbf{R}^n$, for each $i \in \mathbf{N}$ the distance $d(x_i, S)$ can be approximated by the number

$$d(x_i, A_0 \cup \dots \cup A_m). \qquad (4)$$

So, if for $i, k \in \mathbf{N}$ we can effectively compute the number (4), the sequence $(d(x_i, S))$ is computable and S is a recursive set.

The idea of finding sets $A_0, \dots, A_m$ with properties (I)-(III) comes from [4], where this concept is used in the proof of the fact that co-r.e. arc with computable endpoints must be recursive (actually of a more general fact). The essential notion in that proof is the notion of a chain.

A finite sequence $C_0, \dots, C_m$ of nonempty open subsets of $\mathbf{R}^n$ is said to be a **chain** in $\mathbf{R}^n$ if

$$C_i \cap C_j \neq \emptyset \Leftrightarrow |i - j| \le 1,$$

$\forall i, j \in \{0,\dots,m\}$ ([6]). We say that C_i is a link of the chain $C_0,\dots,C_m$, $i \in \{0,\dots,m\}$.

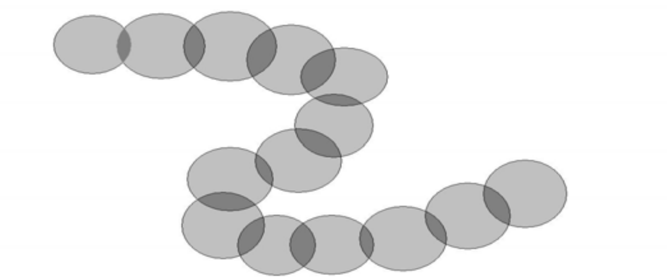

Figure 2. A chain

In general, if $\mathsf{C} = (C_0,\dots,C_m)$ is a finite sequence of subsets of $\mathbf{R}^n$, then we will denote by $\bigcup\mathsf{C}$ the union $C_0 \cup \dots \cup C_m$ and we will say that C covers X, $X \subseteq \mathbf{R}^n$, if $X \subseteq \bigcup\mathsf{C}$. If C_i is nonempty for each $i \in \{0,\dots,m\}$, then we define

$$\operatorname{mesh}\mathsf{C} = \max_{0\le i\le m} \operatorname{diam}(C_i).$$

Now, if S is an arc in $\mathbf{R}^n$ (a continuous injective image of the segment $[0,1]$) with endpoints a and b, then (see [4]):

(i) for each $k \in \mathbf{N}$ there exists a chain $\mathsf{C} = (C_0,\dots,C_m)$ which covers S such that $\operatorname{mesh}\mathsf{C} < 2^{-k}$ and $a \in C_0$, $b \in C_m$;

(ii) if $\mathsf{C} = (C_0,\dots,C_m)$ is a chain which covers S such that $a \in C_0$, $b \in C_m$, then each link of C intersects S.

These two properties are crucial in the proof of the fact that a co-r.e. arc S with computable endpoints a and b must be recursive. Namely, using the fact that S is r.e. and a and b are computable, for each $k \in \mathbf{N}$ we can effectively find some chain C with property (i). This means that properties (I) and (II) are satisfied (for the set S and the finite sequence C), however (III) also holds which follows from (ii).

In our case, in the case of a triod, we modify this proof in the following way. We say that chains $C_0,\dots,C_m$ and $D_0,\dots,D_{m'}$ are **complementary** if for all $i \in \{0,\dots,m\}$, $j \in \{0,\dots,m'\}$ the following implication holds:

$$C_i \cap D_j \neq \emptyset \Rightarrow i = m \text{ and } j = m'.$$

A triple $(\mathsf{C},\mathsf{D},\mathsf{E})$ is called a **T-chain** if C,D and E are chains each two of which are mutually complementary. A T-chain $(\mathsf{C},\mathsf{D},\mathsf{E})$ is said to be an $\varepsilon-$T-chain, where $\varepsilon \in \mathbf{R}$, if $\operatorname{mesh}\mathsf{C} < \varepsilon$, $\operatorname{mesh}\mathsf{D} < \varepsilon$ and $\operatorname{mesh}\mathsf{E} < \varepsilon$. We say that a T-chain $(\mathsf{C},\mathsf{D},\mathsf{E})$ covers X, $X \subseteq \mathbf{R}^n$, if

$$X \subseteq \left(\bigcup\mathsf{C}\right) \cup \left(\bigcup\mathsf{D}\right) \cup \left(\bigcup\mathsf{E}\right).$$

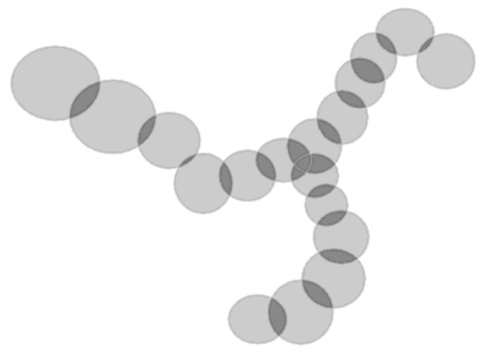

Figure 3. A T-chain

Suppose that S is a co-r.e. triod with computable endpoints a,b and c. We want an effective procedure which, for a given $k \in \mathbf{N}$, gives a triple $(\mathsf{C},\mathsf{D},\mathsf{E})$ with the following properties:

(I*) $(\mathsf{C},\mathsf{D},\mathsf{E})$ is a $2^{-k}-$T-chain;

(II*) $(\mathsf{C},\mathsf{D},\mathsf{E})$ covers S;

(III*) each link of the chains C,D and E intersects S.

Namely, if we have a T-chain with properties (I*), (II*) and (III*) and if $\mathsf{C} = (C_0,\dots,C_m)$, $\mathsf{D} = (D_0,\dots,D_{m'})$, $\mathsf{E} = (E_0,\dots,E_{m''})$, then, for the the set S and the finite sequence of sets $C_0,\dots,C_m,D_0,\dots,D_{m'},E_0,\dots,E_{m''}$ properties (I)-(III) hold. So the existence of an effective procedure which, for a given $k \in \mathbf{N}$, gives a T-chain with properties (I*), (II*) and (III*) implies that S is recursive.

However, such a procedure do exist: in the same way as in case of arcs in [4] we can, for a given $k \in \mathbf{N}$, effectively find a triple $(\mathsf{C},\mathsf{D},\mathsf{E})$ such that

(1*) $(\mathsf{C},\mathsf{D},\mathsf{E})$ is a $2^{-k}-$T-chain;

(2*) $(\mathsf{C},\mathsf{D},\mathsf{E})$ covers S;

(3*) $a \in C_0$, $b \in D_0$ and $c \in E_0$, where C_0, D_0, E_0 are first links of the chains $\mathsf{C}, \mathsf{D}, \mathsf{E}$ respectively. Such a T-chain clearly satisfies properties (I*) and (II*), but it also satisfies property (III*). If we suppose, for example, that there exists some $i \in \{0, \ldots, m\}$ such that $C_i \cap S = \emptyset$, then clearly $i \geq 1$ and, if $i < m$,

$$C_0 \cup \ldots \cup C_{i-1},$$

$$C_{i+1} \cup \ldots \cup C_m \cup \left(\bigcup \mathsf{D}\right) \cup \left(\bigcup \mathsf{E}\right)$$

are two disjoint open sets whose union covers S and each of them intersects S. This is impossible since S, as a triod, is a connected set. Hence $i = m$, but this also, in the same way, gives a contradiction.

III. Formal proof

In this section we will give some technical details which will formalize the idea of the previous section. We will see that the main result of this paper, the result that each co-r.e. triod with computable endpoints is recursive, holds not just in the Euclidean space, but more generally in some computable metric space.

A computable metric space is a tuple (X, d, α), where (X, d) is a metric space and $\alpha : \mathbf{N} \to X$ is a sequence dense in (X, d) such that the function $\mathbf{N}^2 \to \mathbf{R}$, $(i, j) \mapsto d(\alpha_i, \alpha_j)$ is computable. (The notion of a computable function $\mathbf{N}^2 \to \mathbf{R}$ is defined in the same way as the notion of a computable sequence of real numbers.)

Example 2 Let $n \geq 1$ and let (α_i) be some computable sequence in $\mathbf{R}^n$ such that the set $\{\alpha_i \mid i \in \mathbf{N}\}$ is dense in $\mathbf{R}^n$. Then $(\mathbf{R}^n, d, (\alpha_i))$ is a computable metric space, where d is the Euclidean metric.

Let (X, d, α) be a computable metric space. Let (q_i) be some computable sequence of rational numbers (in the sense that $q_i = (-1)^{c(i)} \dfrac{a(i)}{b(i)}$, $\forall i \in \mathbf{N}$, where $a, b, c : \mathbf{N} \to \mathbf{N}$ are recursive functions) such that

$$\{q_i \mid i \in \mathbf{N}\} = \mathbf{Q} \cap \langle 0, \infty \rangle.$$

Let $\tau_1, \tau_2 : \mathbf{N} \to \mathbf{N}$ be some recursive functions such that

$$\{(\tau_1(i), \tau_2(i)) \mid i \in \mathbf{N}\} = \mathbf{N}^2.$$

For $i \in \mathbf{N}$ we define the sets I_i and $\overline{I}_i$ by

$$I_i = B(\alpha_{\tau_1(i)}, q_{\tau_2(i)}), \overline{I}_i = \overline{B}(\alpha_{\tau_1(i)}, q_{\tau_2(i)}).$$

Here, for $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball of radius r centered at x and by $\overline{B}(x, r)$ the corresponding closed ball, i.e.

$B(x, r) = \{y \in X \mid d(x, y) < r\}$,

$\overline{B}(x, r) = \{y \in X \mid d(x, y) \leq r\}$.

The sets I_i represent ``rational balls'' and $\overline{I}_i$ ``closed rational balls'' in (X, d, α). The next step is to find some effective enumeration of all finite unions of rational balls. In order to do this, we first fix some recursive functions $\sigma : \mathbf{N}^2 \to \mathbf{N}$ and $\eta : \mathbf{N} \to \mathbf{N}$ with the following property: each finite sequence $a_0, \ldots, a_m$ in $\mathbf{N}$ is equal

$$\sigma(j, 0), \ldots, \sigma(j, \eta(j))$$

for some $j \in \mathbf{N}$. We are going to use the following notation: $(j)_i$ instead of $\sigma(j, i)$ and $\overline{j}$ instead of $\eta(j)$. Hence for each $m \in \mathbf{N}$ and $a_0, \ldots, a_m \in \mathbf{N}$ there exists $j \in \mathbf{N}$ such that

$$(a_0, \ldots, a_m) = ((j)_0, \ldots, (j)_{\overline{j}}).$$

For $j \in \mathbf{N}$ we define

$$J_j = I_{(j)_0} \cup \ldots \cup I_{(j)_{\overline{j}}}.$$

The sets J_j, $j \in \mathbf{N}$, represent finite unions of rational balls in (X, d, α).

Let (X, d, α) be a computable metric space. We say that $x \in X$ is a **computable point** in (X, d, α) if there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that $d(x, \alpha_{f(k)}) < 2^{-k}$, $\forall k \in \mathbf{N}$.

A closed subset S of (X, d) is said to be **recursively enumerable** in (X, d, α) if

$$\{i \in \mathbf{N} \mid S \cap I_i \neq \emptyset\}$$

is a recursively enumerable subset of $\mathbf{N}$ (in sense of classical recursion theory).

A closed subset S is said to be **co-recursively enumerable** in (X, d, α) if $S = \emptyset$ or there exists a recursive function $f : \mathbf{N} \to \mathbf{N}$ such that

$$X \setminus S = \bigcup_{i \in \mathbf{N}} I_{f(i)}.$$

We say that S is a **recursive** set in (X, d, α) if S is both recursively enumerable and co-recursively enumerable.

Example 3 Let $(\mathbf{R}^n, d, (\alpha_i))$ be the computable metric space of Example 2. Then $x \in \mathbf{R}^n$ is a computable point in this computable metric space if and only if x is a computable point in the sense of Section *I*. Furthermore, S is a co-r.e. set in this computable metric space if and only if S is co-r.e. subset of $\mathbf{R}^n$ in the sense of Section *I* and, similarly, S is recursive in $(\mathbf{R}^n, d, (\alpha_i))$ if and only if S is a recursive subset of $\mathbf{R}^n$.

We say that a computable metric space (X, d, α) has the **effective covering property** ([2]) if

$$\{(i,j) \in \mathbf{N}^2 \mid \overline{I}_i \subseteq J_j\}$$

is a recursively enumerable subset of $\mathbf{N}^2$. For example, the computable metric space $(\mathbf{R}^n, d, (\alpha_i))$ of Example 2 has the effective covering property ([4]).

Theorem 1 *Let (X, d, α) be a computable metric space which has the effective covering property and compact closed balls. Let S be a co-r.e. triod in this space with computable endpoints a, b and c. Then S is recursive.*

Proof. For $l \in \mathbf{N}$ let H_l be the finite sequence of sets

$$J_{(l)_0}, \dots, J_{(l)_{\bar{l}}}.$$

The idea of the proof given in the previous section now gets a precise form: we want an algorithm which, for a given $k \in \mathbf{N}$, gives $l_1, l_2, l_3 \in \mathbf{N}$ such that for the triple $(\mathsf{H}_{l_1}, \mathsf{H}_{l_2}, \mathsf{H}_{l_3})$ properties (1*), (2*) and (3*) hold. In other words, we want to prove that there exist recursive functions $L_1, L_2, L_3 : \mathbf{N} \to \mathbf{N}$ such that for each $k \in \mathbf{N}$ the following holds:

(1) $(\mathsf{H}_{L_1(k)}, \mathsf{H}_{L_2(k)}, \mathsf{H}_{L_3(k)})$ is a 2^{-k} – T-chain;

(2) $(\mathsf{H}_{L_1(k)}, \mathsf{H}_{L_2(k)}, \mathsf{H}_{L_3(k)})$ covers S;

(3) $a \in J_{(L_1(k))_0}$, $b \in J_{(L_2(k))_0}$ and $c \in J_{(L_3(k))_0}$.

First, note the following: for each $k \in \mathbf{N}$ there exist numbers $l_1, l_2, l_3 \in \mathbf{N}$ with described property. Indeed, using the fact that S is a continuous injective image of the set T defined by (3), we can divide S into the parts $C_0, \dots, C_m$, $D_0, \dots, D_{m'}$, $E_0, \dots, E_{m''}$ as in Figure 4, where each of these

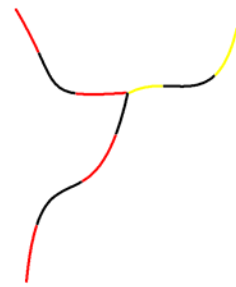

Figure 4.

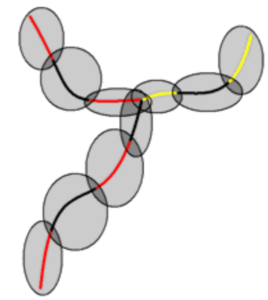

Figure 5.

sets has sufficiently small diameter. Using the fact that these sets are compact, it is easy to conclude that each of them can be replaced by a bigger set of the form J_j for some $j \in \mathbf{N}$ in such way that new sets form a T-chain $(\mathsf{C}, \mathsf{D}, \mathsf{E})$ which satisfies (1*), (2*) and (3*) (Figure 5). Each link in this T-chain is finite union of rational balls and it follows that there exist numbers l_1, l_2, l_3 with desired property.

Using the previous construction and applying the arguments used in the proof of Theorem 35 in [4], we get in a similar way as in [4] that there exist recursive functions $L_1, L_2, L_3 : \mathbf{N} \to \mathbf{N}$ with properties (1)-(3). As noticed before, this gives the recursiveness of S. Formally, we apply Lemma 14 in [4] and the fact that S is recursive follows.

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