An Efficient Algorithm for Reliability Upper Bound of Distributed Systems with Unreliable Nodes

Mohamed H. S. Mohamed, Yang Xiaozong, Liu Hongwei and Wu Zhibo
School of Computer Science and Technology
Harbin Institute of Technology
Harbin 150001, China
mhsm68@yahoo.com

***Abstract*— The reliability of distributed systems in which the communication links are considered reliable while the computing nodes may fail with certain probabilities have been modeled by a probabilistic network or a graph *G*. Computing the residual connectedness reliability (RCR), denoted by *R*(*G*), of probabilistic networks with unreliable nodes is very useful, but is an NP-hard problem. To derive the exact *R*(*G*) expressions for large networks can become rather complex. As network size increases, the reliability bounds could be used to estimate the reliability of the networks. In this paper, we present an efficient algorithm for computing the reliability upper bound of distributed systems with unreliable nodes. We also apply our algorithm to some typical classes of graphs in order to evaluate the upper bound and show the effectiveness and the efficiency of the new algorithm. Numerical results are presented.**

Keywords- distributed systems; residual connectedness reliability; probabilistic network; upper bound

I. INTRODUCTION

There are various reliability problems that occur when a distributed system is modeled by a probabilistic network or a graph G whose nodes and/or edges may fail [1]. The ability of the communication between the residual (remaining working) nodes is measured by the RCR $R(G)$, which is the probability that the residual nodes can communicate with each other [2],[3].

Generally, there are three kinds of fault models in a probabilistic network [1]:

- **Edge fault model:** The nodes of a graph are perfectly reliable, but the edges fail independently with certain probabilities.
- **Node fault model:** The edges of a graph are perfectly reliable, but the nodes fail independently with certain probabilities.
- **Node-and-edge fault model:** Nodes and edges fail independently of each other, with node and edge failure probabilities.

For all these three fault models, it has been shown that the analysis problems are all NP-hard [1],[4]–[6]; that is, there exists no efficient algorithms for computing $R(G)$.

There are quite a number of papers dealing with approximation algorithms for estimating $R(G)$ under the node fault model [7]-[12]. Colbourn [7] proposed a polynomial algorithm of certain restricted classes of graphs, including trees, series-parallel graphs, and permutation graphs. He and Chen [8] developed efficient algorithms of arbitrary graphs, and bound expressions for estimating $R(G)$. They demonstrated theoretically and numerically that the difference between the upper and the lower bounds gradually tends to zero as the number of nodes tends to infinity under the condition that the node failure probability is reasonably low, e.g., less than 0.1. He, Tian and Chen [9] presented a new approach that combines a Monte Carlo simulation scheme and the deterministic bounding approach in [8] to obtain a probabilistic point estimator for $R(G)$. Mohamed, Xiaozong, Hongwei and Zhibo [11] proposed an efficient algorithm for reliability lower bound of distributed systems with unreliable nodes. They showed that their algorithm is faster than the algorithm for the lower bound by He and Chen [8],[9]. They also demonstrated that their lower is tighter than the lower bound in [8],[9]. Mohamed, Xiaozong, Hongwei and Zhibo [12] improved the efficiency and the effectiveness of their algorithm in [11].

Since it may need exponential time of the network size to compute the exact value of $R(G)$, it is important to calculate its tight approximate value at a moderate calculation time.

In this paper, we present a new approach with efficient algorithm for evaluating the reliability upper bound of distributed systems under the node fault model. We apply our algorithm to some typical classes of graphs to evaluate the upper bound and show the effectiveness and the efficiency of the new algorithm. We also demonstrate that the new upper bound is tighter than the upper bound in [8],[9], and is calculated in time $O(n^2)$, where n is the number of nodes in G.

II. NOTATIONS AND ACRONYMS

RCR	Residual Connectedness Reliability
UB	Upper Bound
NF	Node Fault
G	graph
n	number of nodes in G

q	node failure probability
$R(G)$	residual connectedness reliability
$\overline{R}(G)$	upper bound of $R(G)$

III. METHOD

The residual connectedness reliability, $R(G)$, of a probabilistic graph G is the probability that the residual subgraph is connected. We thus have the following definition:

$$R(G) = \Pr\{\text{the subgraph induced by the residual nodes of } G \text{ is connected}\}, \tag{1}$$

where $\Pr\{A\}$ stands for the probability of random event A.

Without loss of generality, we assume that graph G is connected initially and its node set is $V = \{v_1, \ldots, v_n\}$.

Let E denote the random event that the residual nodes induced subgraph is connected, and $\overline{E}$ is the complement event of E. then, according to formula (1), we have

$$R(G) = \Pr\{E\} = 1 - \Pr\{\overline{E}\}. \tag{2}$$

Then the UB of reliability $R(G)$ is given in Theorem 1.

Theorem 1: Let $r = |S|$, $S \in V$, and $S = \{u_1, \ldots, u_r\}$, where any two nodes of S have not any common neighbor node in G, and the nodes of any pair of adjacent nodes in S have the same degree in G. Then the UB of $R(G)$ is

$$R(G) \le \prod_{i=1}^{r}\left(1-(1-q)q^{f_i}\right), \tag{3}$$

where q is the node failure probability, $f_i = |N_G(u_i)|$, and $N_G(u_i)$ is the neighboring set of nodes u_i in G.

Proof: Let F_i be the event that u_i is isolated in the residual subgraph of G, $i = 1, \ldots, r$. Then

$$\bigcup_{i=1}^{r} F_i \subseteq \overline{E} \tag{4}$$

and

$$R(G) \le 1 - \Pr\left\{\bigcup_{i=1}^{r} F_i\right\}. \tag{5}$$

It is obvious that F_i occurs if and only if u_i is operating and all its neighbor nodes fail. The probability that such an event occurs is

$$\Pr\{F_i\} = (1-q)q^{f_i},\ i = 1, \ldots, r. \tag{6}$$

Let u_j and u_k be a pair of nodes in S and $d_{j,k}$ be the distance between u_j and u_k in G. Then since u_j and u_k do not have any common neighbor in G, it is clear that $d_{j,k} \ne 2$. That is, $d_{j,k}$ is either 1 or at least 3. Consequently, for u_j and u_k with $d_{j,k} \ge 3$, it is clear that $F_j \cap F_k = \varnothing$. Suppose that u_j and u_k with $d_{j,k} = 1$, then from the definition of F_j and F_k, it is obvious that u_j is operating in F_j but is failed in F_k, while u_k is operating in F_k but is failed in F_j, and since u_j and u_k do not have any common neighbor in G, we have $F_j \cap F_k = \varnothing$.

Therefore

$$\Pr\left\{\bigcup_{i=1}^{r} F_i\right\} = 1 - \prod_{i=1}^{r} \Pr\{\overline{F}_i\}. \tag{7}$$

Thus from (5), (6) and (7), we get

$$R(G) \le \prod_{i=1}^{r}\left(1-(1-q)q^{f_i}\right).$$

Now the question is how tight is this bound with respect to the number of nodes in S. From the proof of the theorem we can see that the more nodes are included in S, the tighter the UB will be. Thus, in order to obtain a tighter UB, we must find the set S with as many nodes as possible.

IV. ALGORITHM FOR THE UPPER BOUND

Based on the constructive method for estimating the UB above, we can obtain an algorithm which estimates the reliability UB. According to Theorem 1, the bigger the set S, the best the UB is.

A. Preparation

Before we derive the algorithm, we need the following procedure to find the set S.

Procedure $S(G)$
Input: graph G
Output: a set S of graph G

$S \leftarrow \varnothing$, $H \leftarrow V$, $i \leftarrow 0$
while $H \ne \varnothing$
  do $i \leftarrow i+1$
    $u_i \leftarrow$ the node in H with min degree in G
    $S \leftarrow S \cup \{u_i\}$
    $neig \leftarrow 0$
    for each node $v \in N_G(u_i)$
      do if $neig = 0$
        then if v and u_i have the same degree and do not have any common neighbor in G
          then remove v, u_i and their neighbors and neighbors of the neighbors from H
            $i \leftarrow i+1$
            $u_i \leftarrow v$
            $S \leftarrow S \cup \{u_i\}$
            $neig \leftarrow 1$

if $neig = 1$
then remove u_i and its neighbors and neighbors of the neighbors from H
return S

B. Algorithm

The following algorithm finds the UB $\overline{R}(G)$ of $R(G)$.

Algorithm Upper-Bound
Input: graph G, the node failure probability q
Output: $\overline{R}(G)$ /* $\overline{R}(G)$ according to Theorem 1*/

$\overline{R}(G) \leftarrow 1$
$S \leftarrow S(G)$
for every node u_i of S
do $\overline{R}(G) \leftarrow \overline{R}(G) \times \left(1-(1-q)q^{f_i}\right)$
return $\overline{R}(G)$

V. EXAMPLE

To demonstrate our method algorithm, the sample network by Fig. 1 is considered.

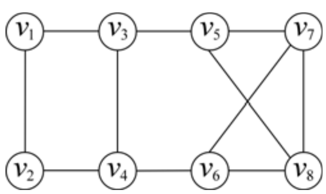

Fig.1 Sample network

For the sample network in Fig. 1, it is clear that

$$V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\};$$

$$N_G(v_1) = \{v_2, v_3\},\ f_1 = 2;$$
$$N_G(v_2) = \{v_1, v_4\},\ f_2 = 2;$$
$$N_G(v_3) = \{v_1, v_4, v_5\},\ f_3 = 3;$$
$$N_G(v_4) = \{v_2, v_3, v_6\},\ f_4 = 3;$$
$$N_G(v_5) = \{v_3, v_7, v_8\},\ f_5 = 3;$$
$$N_G(v_6) = \{v_4, v_7, v_8\},\ f_6 = 3;$$
$$N_G(v_7) = \{v_5, v_6, v_8\},\ f_7 = 3;$$
$$N_G(v_8) = \{v_5, v_6, v_7\},\ f_8 = 3.$$

$F_1 = \{v_1', v_2'', v_3''\}$, $F_2 = \{v_2', v_1'', v_4''\}$, $F_3 = \{v_3', v_1'', v_4'', v_5''\}$, $F_4 = \{v_4', v_2'', v_3'', v_6''\}$, $F_5 = \{v_5', v_3'', v_7'', v_8''\}$, $F_6 = \{v_6', v_4'', v_7'', v_8'',\}$, $F_7 = \{v_7', v_5'', v_6'', v_8''\}$, $F_8 = \{v_8', v_5'', v_6'', v_7''\}$, where v_i' means that node v_i is operational and v_i'' means that v_i is failed.

Applying our method and algorithm to the sample network in Fig.1, we obtain that

$$S = \{v_1, v_2, v_7, v_8\} \text{ and}$$

$$\overline{R}(G) = \left(1-(1-q)q^{f_1}\right)\left(1-(1-q)q^{f_2}\right)\times\left(1-(1-q)q^{f_7}\right)\left(1-(1-q)q^{f_8}\right)$$
$$= \left(1-\left(1-q^2\right)\right)^2\left(1-\left(1-q^3\right)\right)^2.$$

VI. THE COMPLEXITY OF THE BOUND

For computing the UB, the main computational process is to find a set S in the procedure $S(G)$. This procedure can be done in time $O(n^2)$, while the calculation of the bound value, taken directly from the Theorem 1, can be done in time $O(n)$. Thus the complete algorithm for UB takes $O(n^2)$ operations.

VII. COMPUTATIONAL RESULTS AND DISCUSSION

To show the effectiveness and efficiency of the new UB, we apply our algorithm to some typical classes of graphs such as hypercube, circle and Harary graph, simply because these structures allow simpler routing algorithms, higher fault-tolerance ability and reliability. For example, in a p-dimension hypercube, or p-hypercube for short, denoted by Q_p, a large number of computing nodes (2^p nodes) are connected using a smaller number of communication links (p links per node, instead of $2^p - 1$ links per node as required by a complete graph) while keeping a minimal communication delay between the nodes. The hypercube has a symmetric and regular topology, which is very easy to understand and utilize.

To computationally examine the effectiveness and the efficiency of the new UB four sets of graphs were used (see tables 1–4). When compared to the UB by He and Chen [8,9], in each instance the new UB of this paper was tighter.

TABLE 1 UPPER BOUNDS FOR PATH GRAPH P_n

n	***q***	**UB by He [8,9]**	**New UB**
64	0.1	0.691124	0.631382
	0.01	0.978359	0.977391
	0.001	0.997983	0.997973
256	0.1	0.387498	0.265072
	0.01	0.972179	0.968145
	0.001	0.997919	0.997877
1024	0.1	0.038294	0.008235
	0.01	0.947849	0.932029
	0.001	0.997664	0.997495

TABLE 2 UPPER BOUNDS FOR CIRCLE GRAPH C_n

n	***q***	**UB by He [8,9]**	**New UB**
64	0.1	0.827079	0.748785
	0.01	0.997923	0.996837

	0.001	0.999979	0.999968
256	0.1	0.463725	0.31436
	0.01	0.99162	0.987407
	0.001	0.999915	0.999872
1024	0.1	0.045826	0.009766
	0.01	0.966803	0.950573
	0.001	0.999659	0.999489

TABLE 3 UPPER BOUNDS FOR HYPERCUBE Q_p WITH $P = 4$

q	**UB by He [8,9]**	**New UB**
0.1	0.99982	0.99964
0.08	0.999925	0.999849
0.06	0.999976	0.0.999951
0.04	0.999995	0.99999

TABLE 4 UPPER BOUNDS FOR HARRARY GRAPH $H_{n,k}$ WITH $n = 16$ & $k = 3$

q	**UB by He [8,9]**	**New UB**
0.1	0.99982	0.99964
0.08	0.999925	0.999849
0.06	0.999976	0.0.999951
0.04	0.999995	0.99999

As a result, our algorithm is very efficient and can easily be implemented for evaluating reliability UB for distributed systems with unreliable nodes. The new algorithm produce a good approximation for RCR that can be used in general study in graphs and computer networks.

VIII. CONCLUSIONS

In this paper, a new approach with efficient algorithm for evaluating the reliability upper bound of distributed systems with unreliable nodes has been presented. We have applied our method to several typical classes of graphs (networks) to show the effectiveness and the efficiency of the new algorithm. Our approach produces a good evaluation for RCR that can be used in general study in graphs and computer networks.

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